Main Steps

The 4 main steps for proving a language $A$ is not regular is as follows:

Step 1: Demon Picks $k \geq 0$. You are given some pumping length $k \geq 1$.

Step 2: You pick $xyz$. Select $x, y, z$ such that $xyz \in A$ and $|y| \geq k$.

Step 3: Demon Picks Decomposition $u, v, w$. The demon picks $u, v, w$ such that $y = u, v, w$ and $v \neq \epsilon$.

Step 4: You pick $i \geq 0$. Construct a string $xuv^iwz$ that is not in $A$, for some $i \geq 0$. Remember that you may want to set $i$ to 0 in order to accomplish this.

Comments

- The pumping lemma states a property of regular languages. You cannot use it to prove a language is regular, but you can use its contrapositive to prove a language is not regular.

  As a reminder, here is the pumping lemma in its positive form:

  If a language $A$ is regular, then there exists a $k \geq 1$ such that for all strings $x, y, z$ with $xyz \in A$ and $y \geq k$, there exist strings $u, v, w$ with $y = uvw$ and $v \neq \epsilon$, and for all $i \geq 0$, $xuv^iwz \in A$.

- Be sure your string $xyz$ is in $A$.

- Be sure to handle all possible decompositions of the string $y$ as $uvw$. The demon is picking this decomposition, and you cannot pick which decomposition he chooses.

- Don’t choose an $i$ that is fractional or negative! This is not allowed by the statement of the pumping lemma; $i$ must be an integer $\geq 0$.

- Your string $xyz$ should somehow depend on the pumping length $k$. If it doesn’t depend on $k$, then you cannot guarantee that it will be long enough for all possible values the demon provides.
Example

Consider the language \( A = \{ a^n b^n \mid n \geq 0 \} \). We will show that \( A \) is not regular using the pumping lemma.

**Proof.** By way of contradiction, suppose that \( A \) is regular. Then there is some pumping length \( k \geq 0 \) chosen by the demon. Now we take \( x = \epsilon \), \( y = a^k \), and \( z = b^k \). Then \( xyz = a^k b^k \in A \), and \( |y| = k \). The demon must now pick \( u, v, w \) such that \( y = uvw \) and \( v \neq \epsilon \). Say the demon picks \( u, v, w \) of lengths \( j, m, n \), respectively. Then \( k = j + m + n \) and \( m > 0 \). But whatever the demon picks, we can win by taking \( i = 2 \):

\[
xuv^2wz = a^j a^{2m} a^n b^k = a^k a^m k a^k,
\]

which is not in \( A \) because there are different numbers of \( a \)'s and \( b \)'s.