Structural induction is used to prove a property $P$ of all the elements of some recursively-defined data type. The following are some examples of recursively defined data types:

**Example 1:** An *even number* $e$ is either

1. 0,
2. $2 + k$, where $k$ is an even number, or
3. $-k$, where $k$ is an even number.

**Example 2:** A *rooted binary tree* $T$ is either

1. a single root node, or
2. a root node attached to $v_ℓ$ and $v_r$, where $v_ℓ$ and $v_r$ are the roots of rooted binary trees $T_ℓ$ and $T_r$, respectively.

**Example 3:** A *fully bracketed arithmetic expression in* $x$ is a string $E$ over the alphabet $\{[, +, -, *, x}\}$ that is either

1. the symbols 0, 1, or $x$,
2. $[e + e']$, where $e$ and $e'$ are fully bracketed arithmetic expressions, or
3. $[e \ast e']$, where $e$ and $e'$ are fully bracketed arithmetic expressions.

A proof by structural induction consists of two main steps:

1. Prove $P$ for the “base cases” of the definition.
2. Prove $P$ for the result of any recursive combination rule, assuming that it is true for all the parts.

For example, structural induction on the rooted binary tree of example 2 takes the form:

*Proof.* To prove $P(T)$ holds $\forall$ rooted binary trees $T$, show:

**Base** ($T =$ single root node). Show $P$(single node) holds.

**Inductive Step** ($T =$ root node attached to root of subtrees $T_r$ and $T_ℓ$). Assume $P(T_ℓ)$ and $P(T_r)$ in order to prove $P(T)$.

Here’s an actual example:

**Theorem 1.** *For any rooted binary tree $T = (V, E)$, $|E| = |V| - 1$.*

*Proof.* We prove this by structural induction on $T = (V_T, E_T)$. Let $P(T)$ be the property that $|E_T| = |V_T| - 1$.

As a base case, consider when $T$ is a single root node. Then $|E_T| = 0 = 1 - 1 = |V_T| - 1$.

For the inductive step, suppose $T$ consists of a root node $r$, rooted binary trees $T_ℓ$ and $T_r$, and edges $(r, v_ℓ)$ and $(r, v_r)$, where $v_ℓ$ and $v_r$ are the roots of $T_ℓ$ and $T_r$, respectively. Suppose by
way of induction that \( P(T_l) \) and \( P(T_r) \) are both true. That is, suppose that \( |E_l| = |V_l| - 1 \) and \( |E_r| = |V_r| - 1 \). Then
\[
|E_T| = |E_r| + |E_l| + 2 \quad \text{by definition of } T \\
= |V_r| - 1 + |V_l| - 1 + 2 \quad \text{by the IH} \\
= |V_T| - 1 \quad \text{by definition of } T.
\]
Therefore, for every rooted binary tree, the property \( P \) holds.

\[
\text{Theorem 2. The set of even numbers defined in example 1 are elements of } \{2x \mid x \in \mathbb{Z}\}.
\]

\[
\text{Proof.} \quad \text{We prove this by structural induction on even number } e. \quad \text{Let } P(e) \text{ be the property that } e = 2x \text{ for some } x \in \mathbb{Z}.
\]
As a base case, consider when \( e = 0 \). Then \( e = 0 = 2 \cdot 0 \), and \( 0 \in \mathbb{Z} \).

For the inductive step, suppose we have two cases.

First suppose that \( e = e' + 2 \), where \( e' \) is an even number and \( P(e') \) holds. Then
\[
e = e' + 2 \\
= 2x' + 2, \text{ for some } x' \in \mathbb{Z} \quad \text{by the induction hypothesis} \\
= 2(x' + 1) \\
= 2x, \text{ for some } x \in \mathbb{Z} \quad \text{since } x = x' + 1 \in \mathbb{Z}
\]

For the second case, suppose that \( e = -e' \), where \( e' \) is an even number and \( P(e') \) holds. Then
\[
e = -e' \\
= -(2x'), \text{ for some } x' \in \mathbb{Z} \quad \text{by the induction hypothesis} \\
= 2(-x') \\
= 2x, \text{ for some } x \in \mathbb{Z} \quad \text{since } x = -x' \in \mathbb{Z}
\]
Therefore, for every even number \( e \), the property \( P \) holds.

\[
\text{Comments}
\]
- Although many proofs by structural induction have an equivalent proof by mathematical induction (for example, we could prove Theorem 1 by induction on the height of the tree \( T \)), there are structural induction proofs (namely, those on infinite data objects) that are actually strictly more powerful than ordinary induction.
- Structural induction will make your life easier in this course, so try to pick it up early on!

\[
\text{Exercises}
\]
1. Define the natural numbers (i.e. \( \mathbb{Z}^{\geq 0} \)) recursively.
2. Show that if you can prove property \( P(n) \) for all \( n \geq 0 \) by mathematical induction, then you can prove \( P(n) \) for all \( n \geq 0 \) by structural induction. (Hint: use your definition from question 1.)