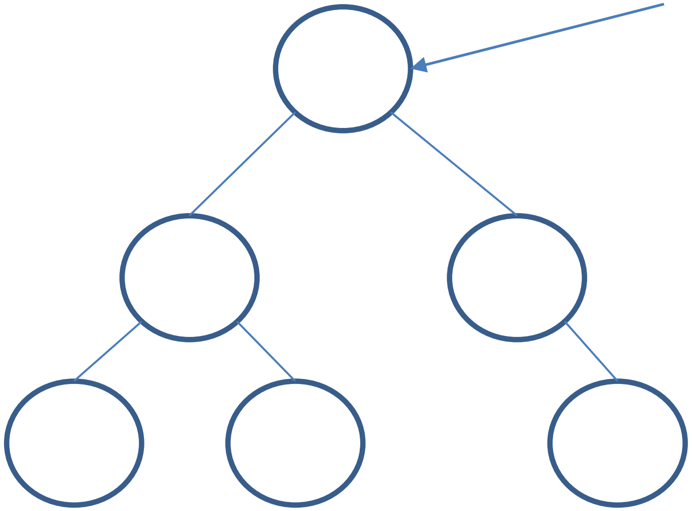
Lab 6. Level count

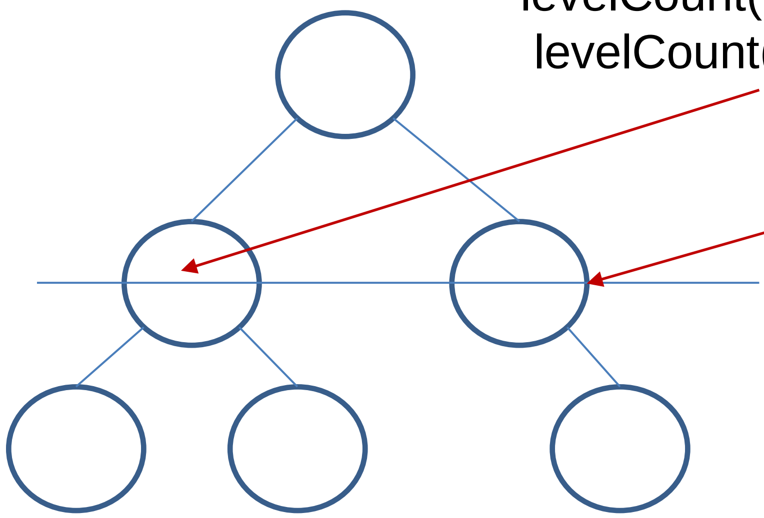
Each tree object can
count nodes at all levels

levelCount(0) returns 1



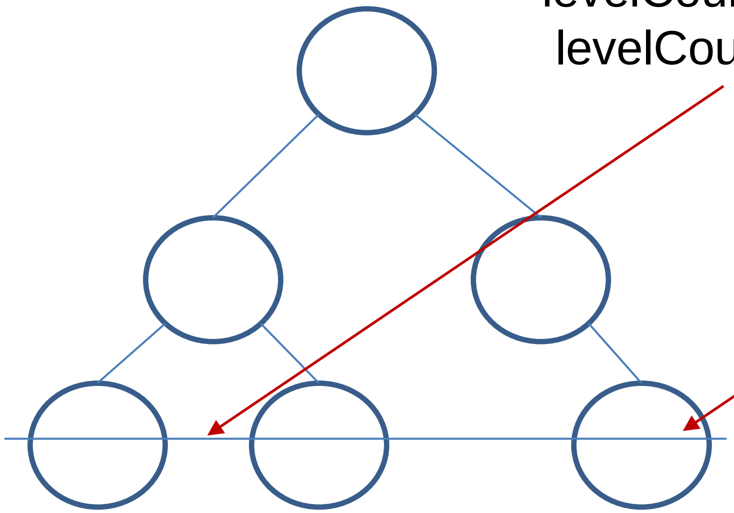
Lab 6. Level count

levelCount(1) returns:
levelCount(0) of leftChild + levelCount(0) of right child

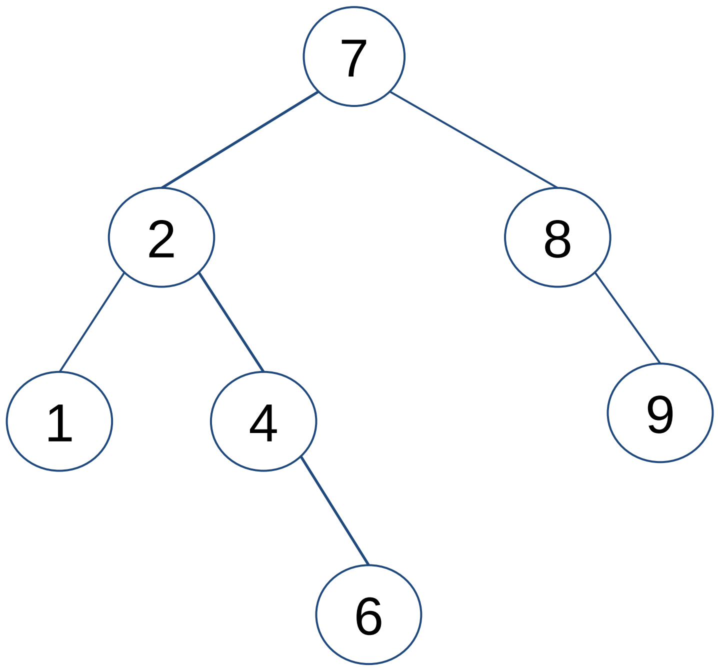


Lab 6. Level count

levelCount(2) returns:
levelCount(1) of leftChild + levelCount(1) of right child



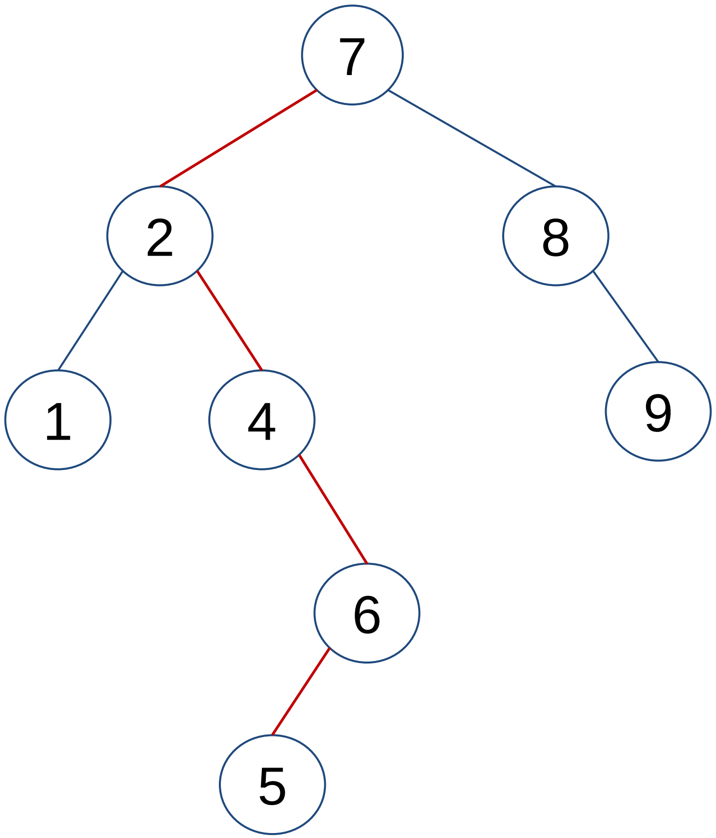
Lab 7. AVL trees



Is this tree balanced?

A. Yes
B. No

Lab 7. AVL trees

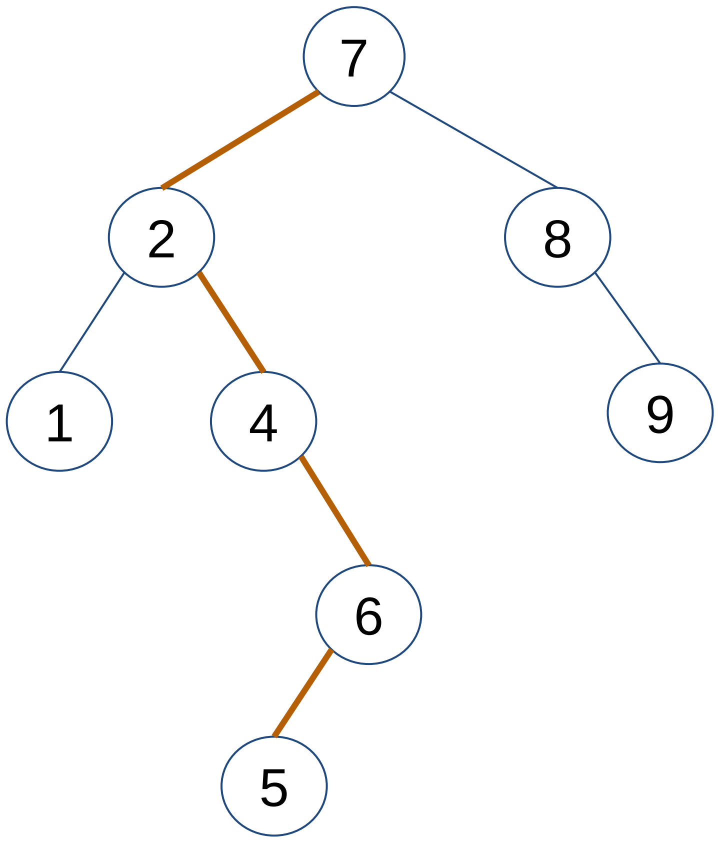


We insert node 5

Is the tree still balanced?

A. Yes
B. No

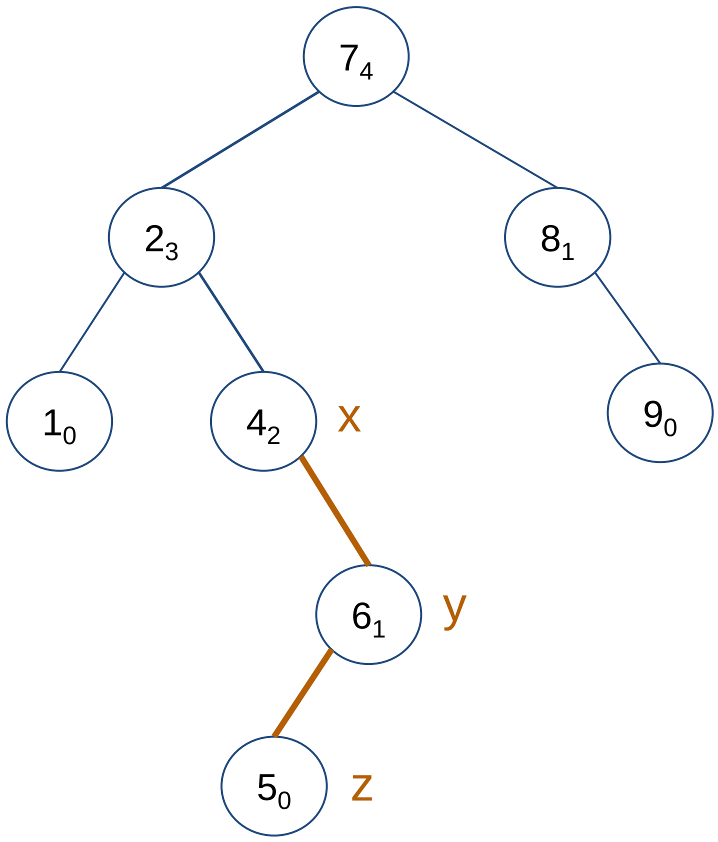
Lab 7. AVL trees



Which node is unbalanced (needs rotation)?

A. Node 6
B. Node 4
C. Node 2
D. Node 7
E. None of the above

Lab 7. AVL trees



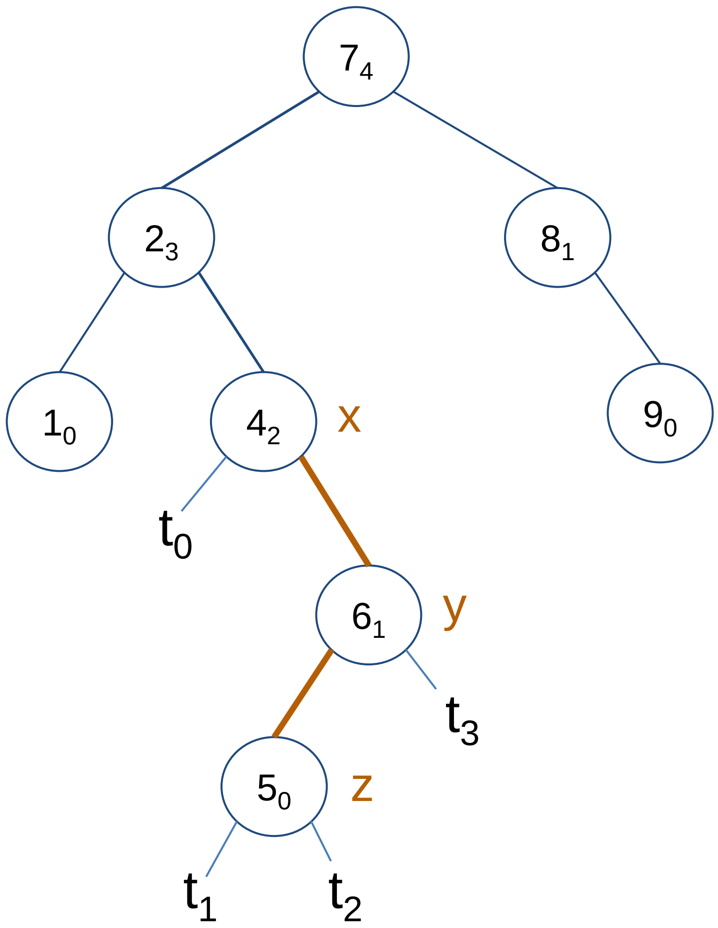
The rebalancing algorithm in the lab is similar to the algorithm we learned in class

Going up from the point of insertion, you find the first unbalanced node and from it you collect its child and its grandchild, moving always into the child with the larger height:

$4 \to 6 \to 5$

$x \to y \to z$

Lab 7. AVL trees



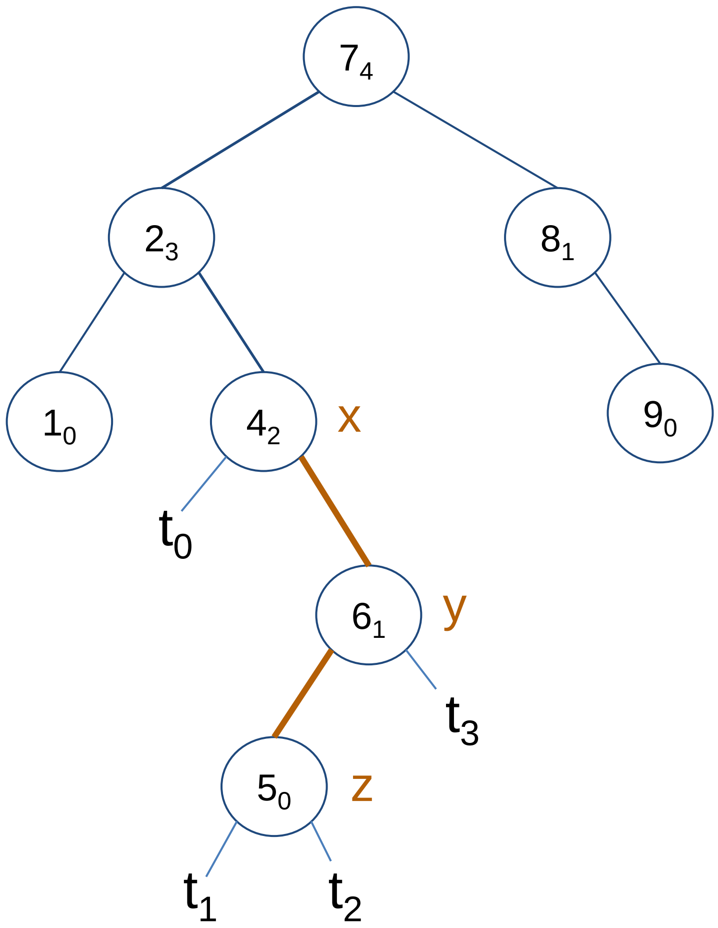
$4 \rightarrow 6 \rightarrow 5$
$x \rightarrow y \rightarrow z$

Now you need to consider 4 different cases:

Case 1: right-left heavy
Determine order for this case:
a=x, b=z, c=y, a<b<c

Collect child subtrees to be reattached:
t_0 = x.left, t_1= z.left, t_2=z.right, t_3=y.right

Lab 7. AVL trees

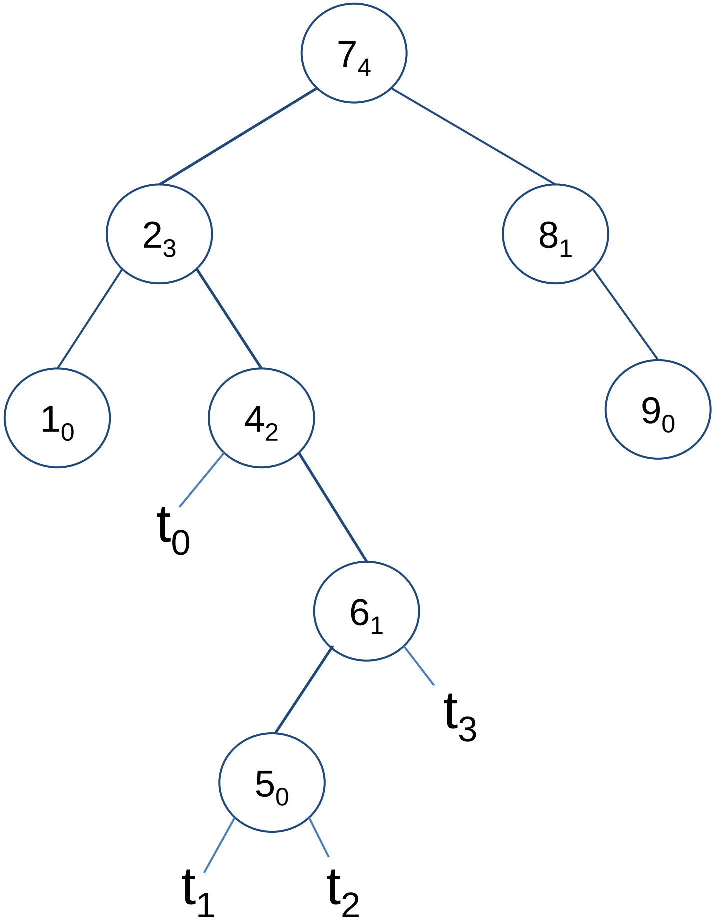


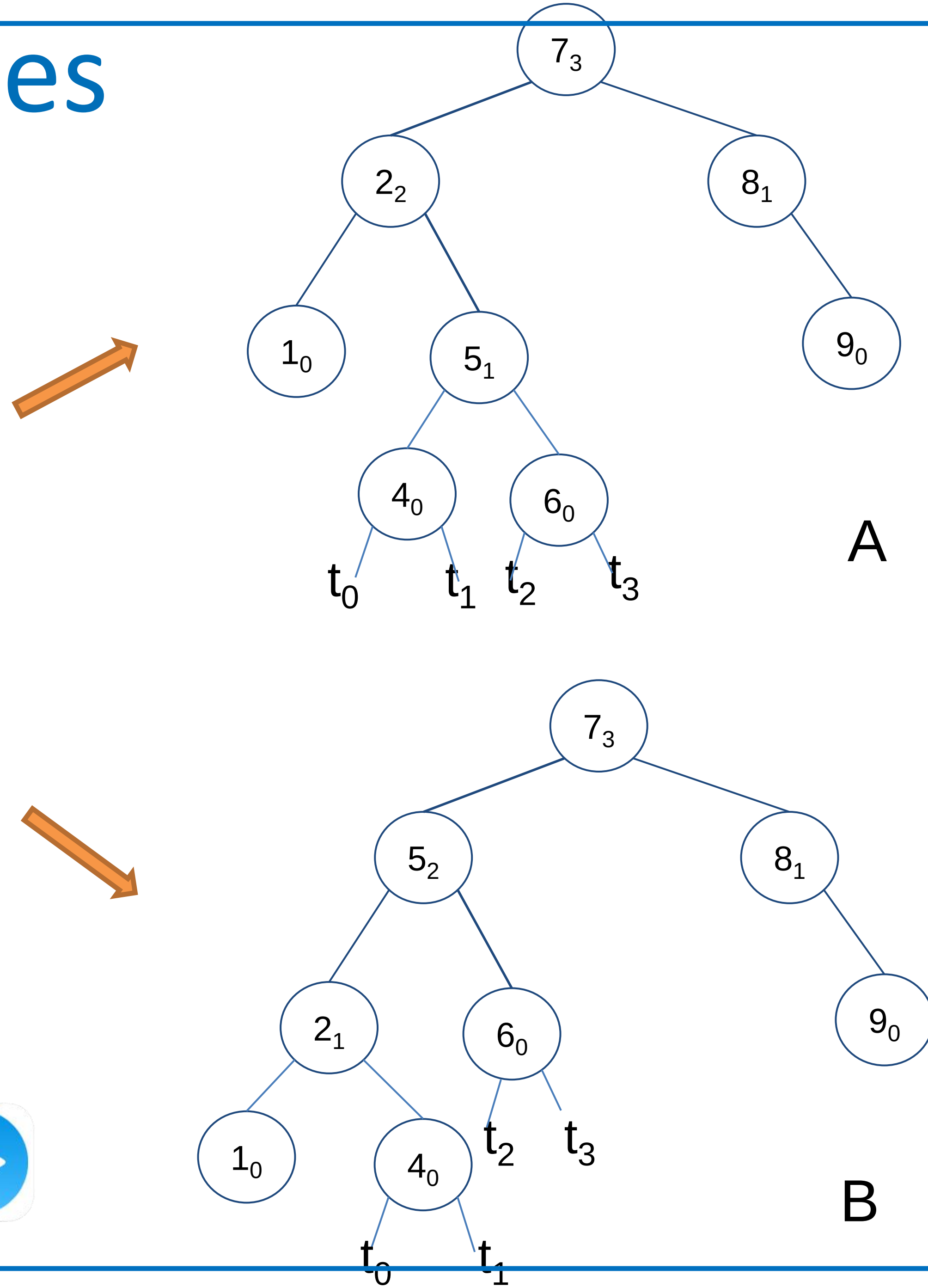
Restructuring is implemented: you only need to consider all the cases and identify 3 nodes and 4 children

a=x, b=z, c=y, a<b<c

t_0 = x.left, t_1= z.left, t_2=z.right, t_3=y.right

Lab 7. AVL trees





Tree after restructuring:
A.
B.
C. None of the above

Priority Queue ADT
Binary heaps

Lecture 22

by Marina Barsky

Priority Queue ADT

- A ***Priority Queue*** is a generalization of a *Queue* where each element is assigned a priority and elements come out in order of priority
- If the priority is the earliest time they were added to the queue then Priority Queue becomes a regular FIFO Queue
- We are interested in a case when priority of each element is not related to the time when the element was added to the queue

Priority Queue ADT

Specification

Priority Queue is an **Abstract Data Type** supporting the following main operations:

- ➔ ***top()*** **-** get an element with the highest priority
- ➔ ***enqueue(e,p)**** - adds a new element *e* with priority *p*
- ➔ ***dequeue()*** **-** removes and returns the element with the highest priority

---

*To simplify the discussion we use *enqueue(p)*, where *p* is a number which reflects the priority

Priority Queue: possible Data Structures

	enqueue	dequeue
Unsorted array/list	O(1)	O(n)
Sorted array/list	O(n)	O(1)

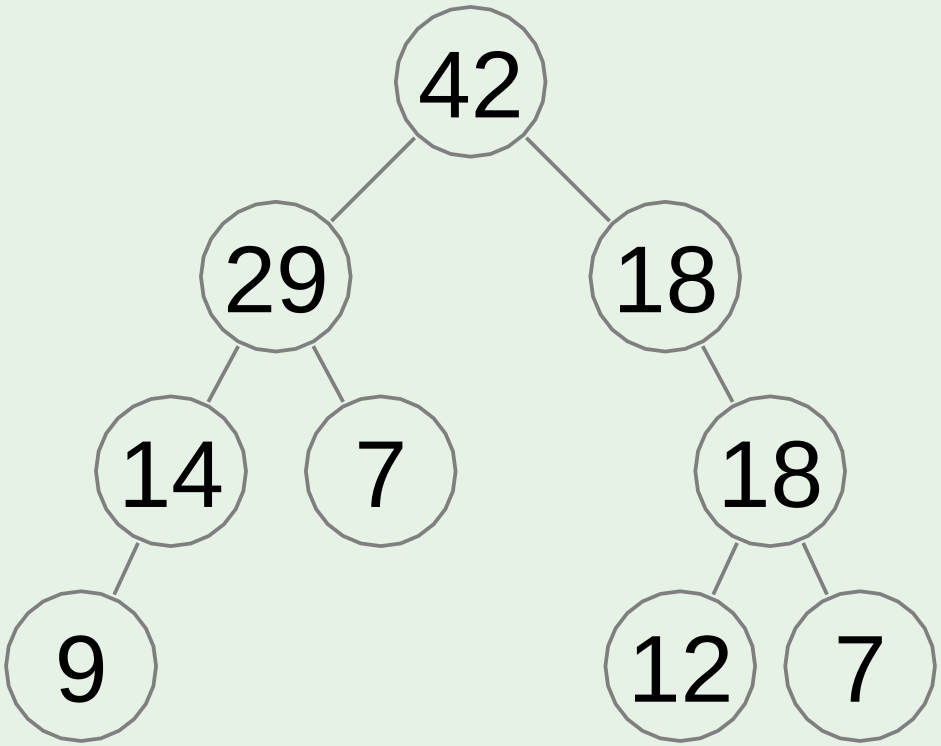
Binary max-heap

Definition

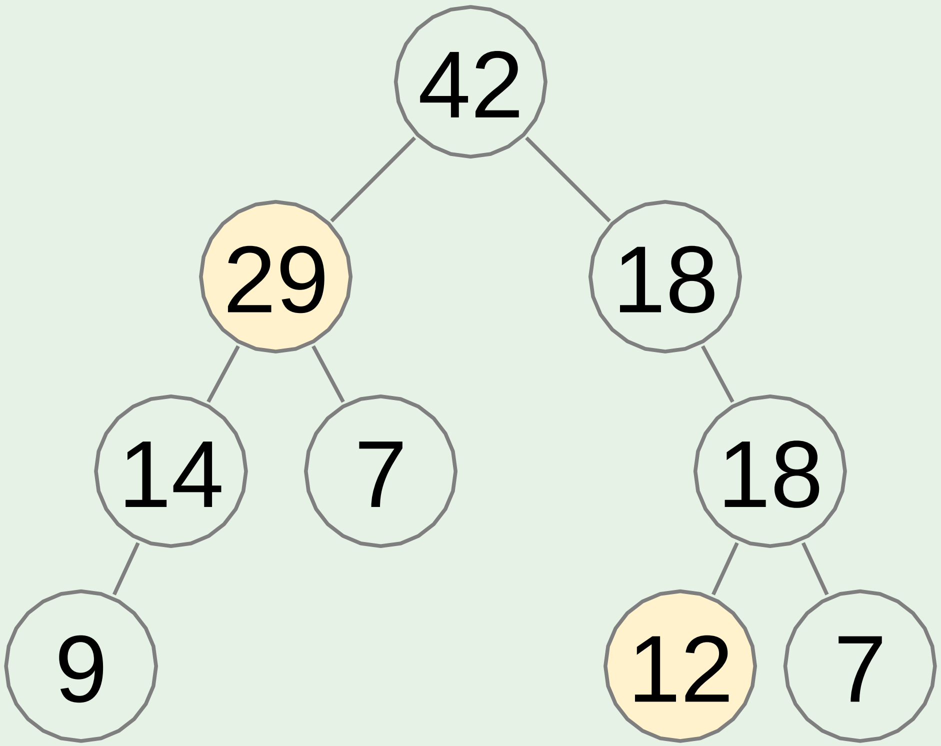
Binary max-heap is a **binary** tree where the value of each node is at least (≥) the values of its children.

https://visualgo.net/en/heap?slide=1

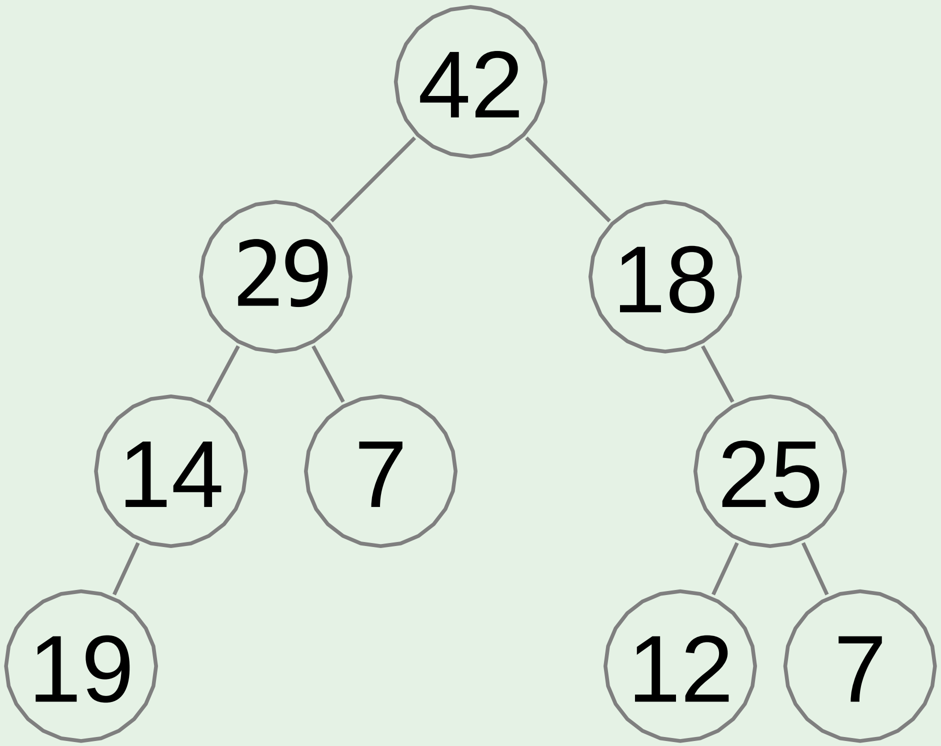
Heap?



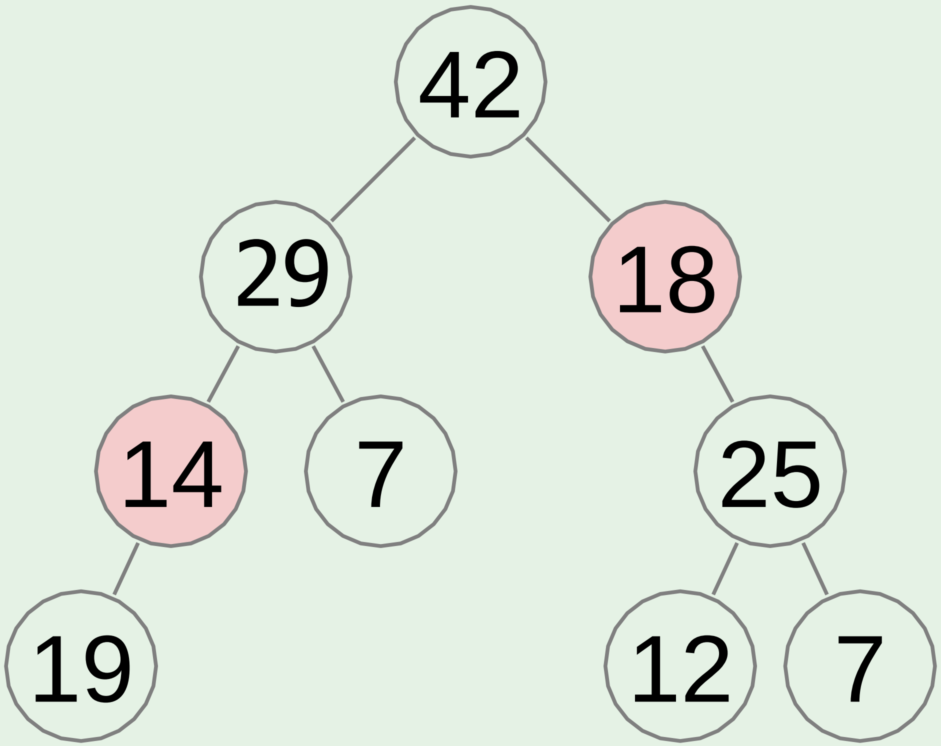
Heap? Yes



Heap?

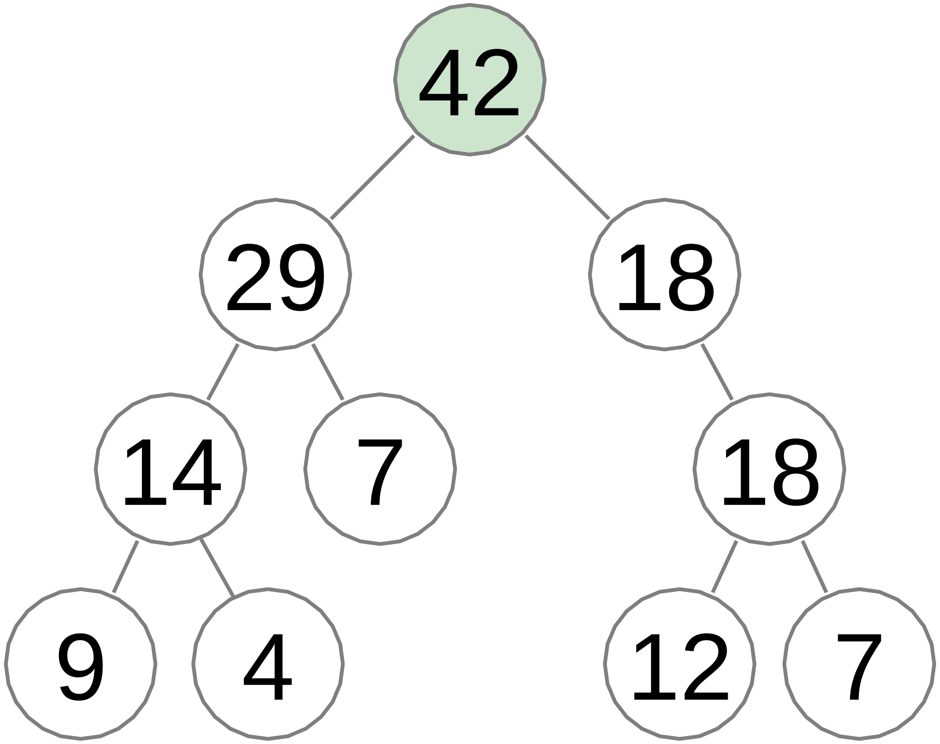


Heap? No



Heap operations: `top`

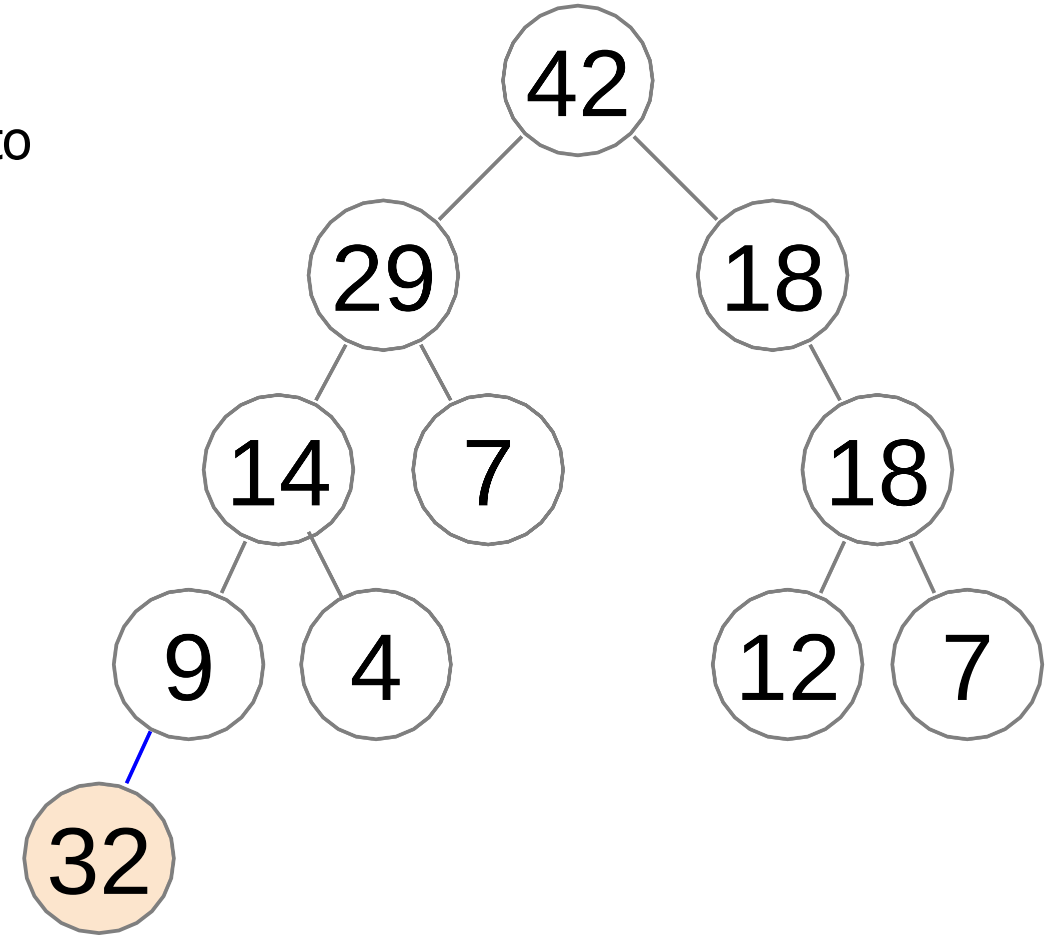
return the root value



Run-time: O(1)

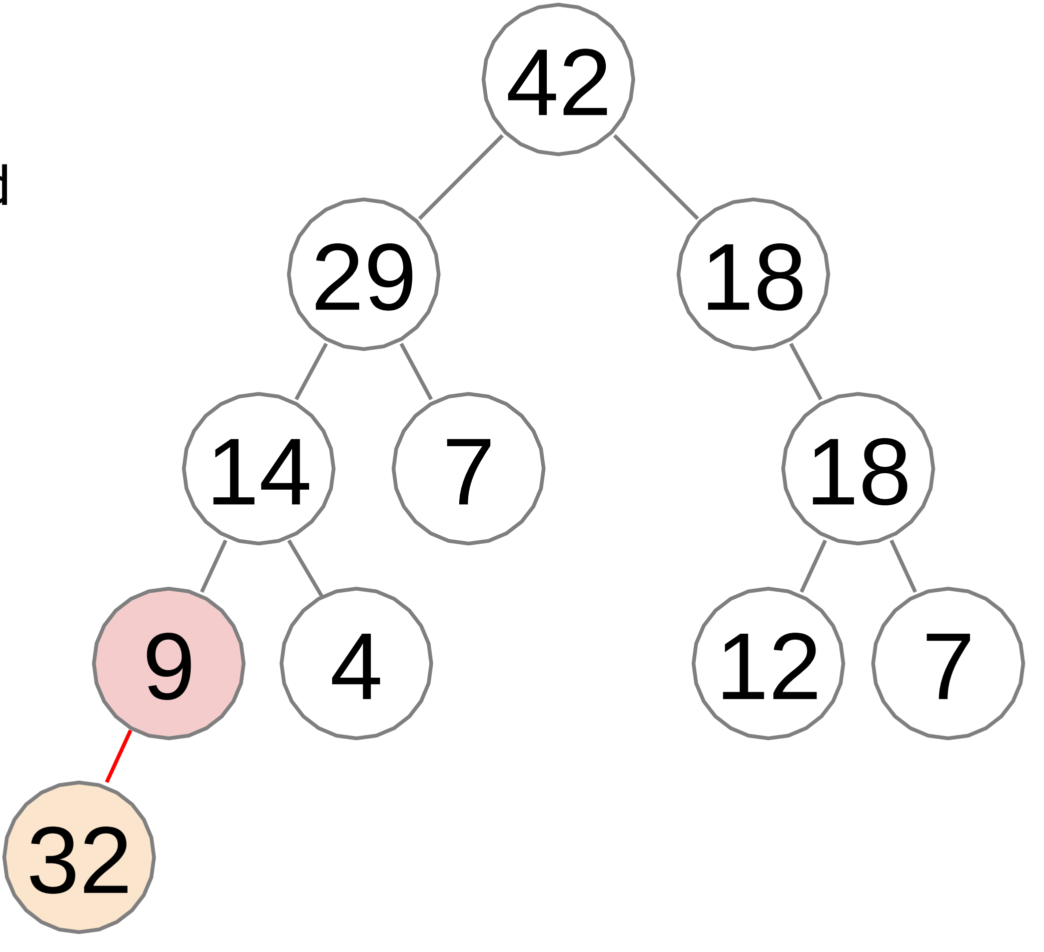
Heap operations: `enqueue(e)`

attach a new node to
any leaf



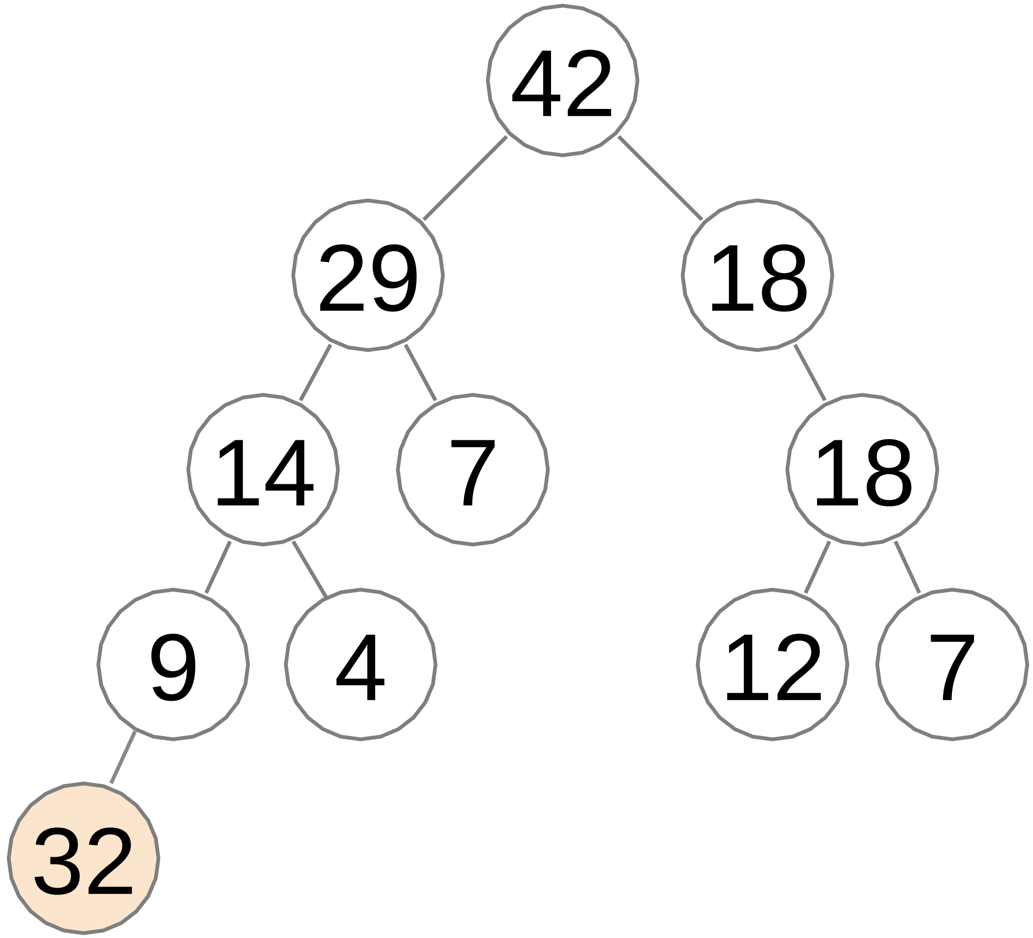
Heap operations: ***enqueue(e)***

the heap property
may become violated



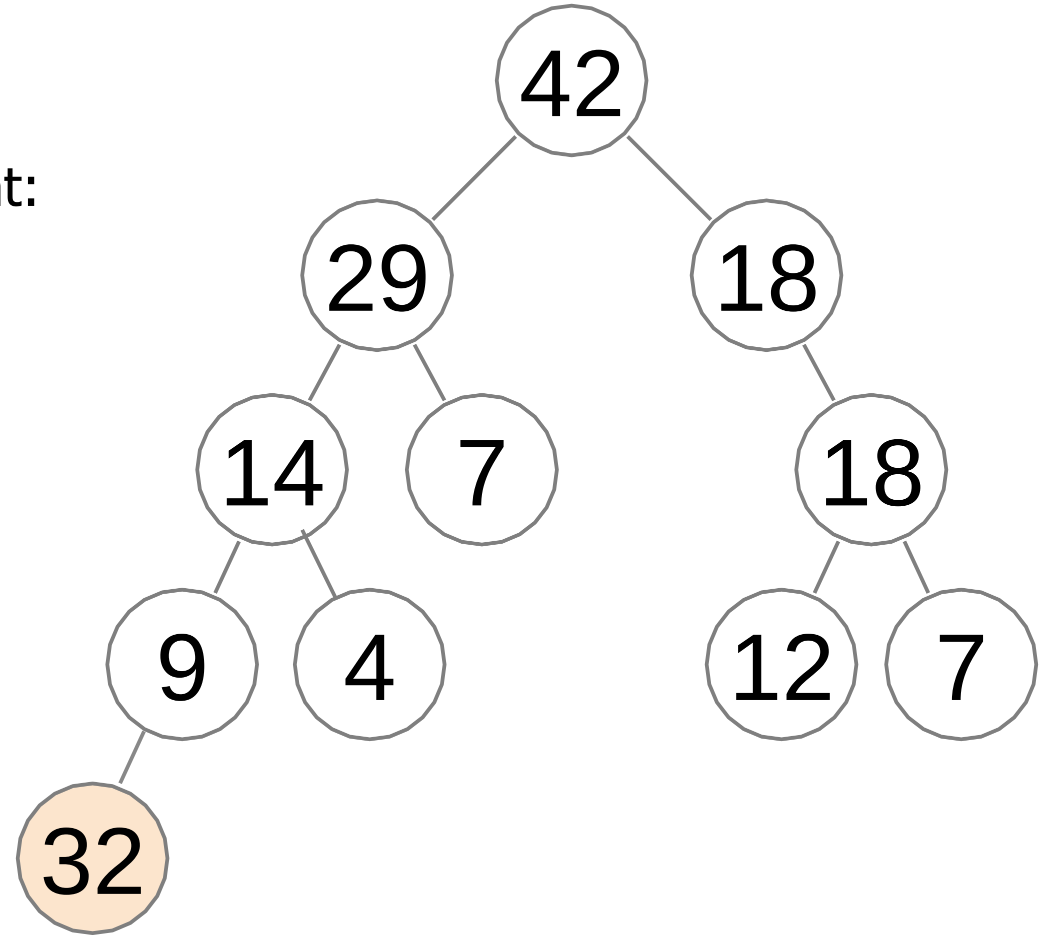
Heap operations: **`enqueue(e)`**

to fix that we let the new node *sift up*



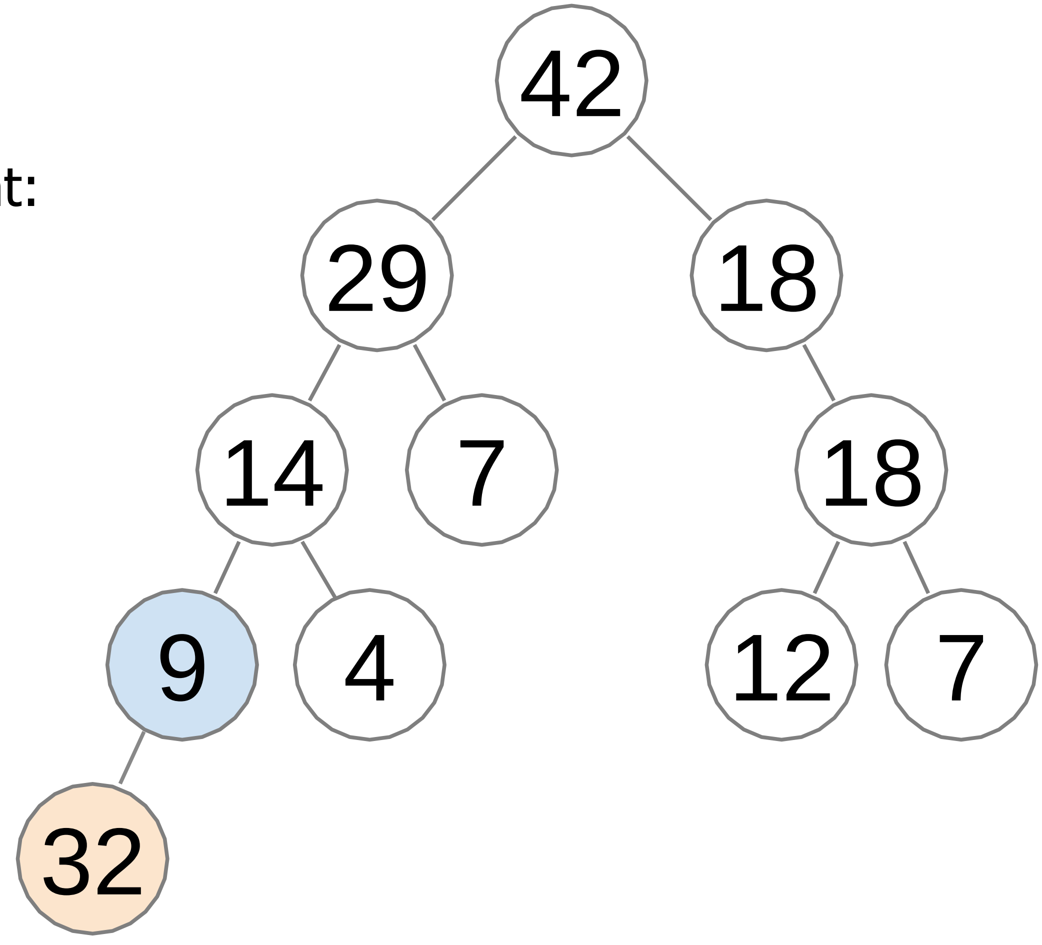
Heap operations: `sift_up(e)`

if current element is
bigger than the parent:
swap



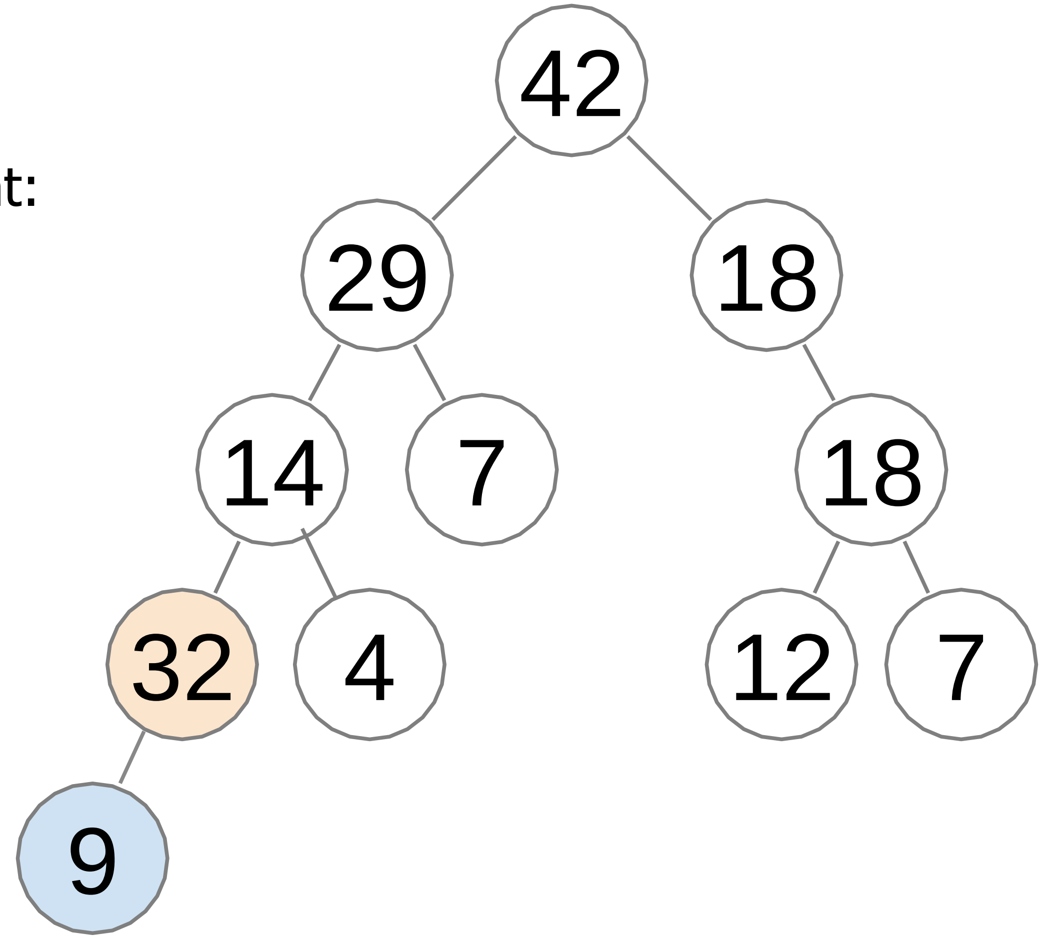
Heap operations: `sift_up(e)`

if current element is
bigger than the parent:
swap



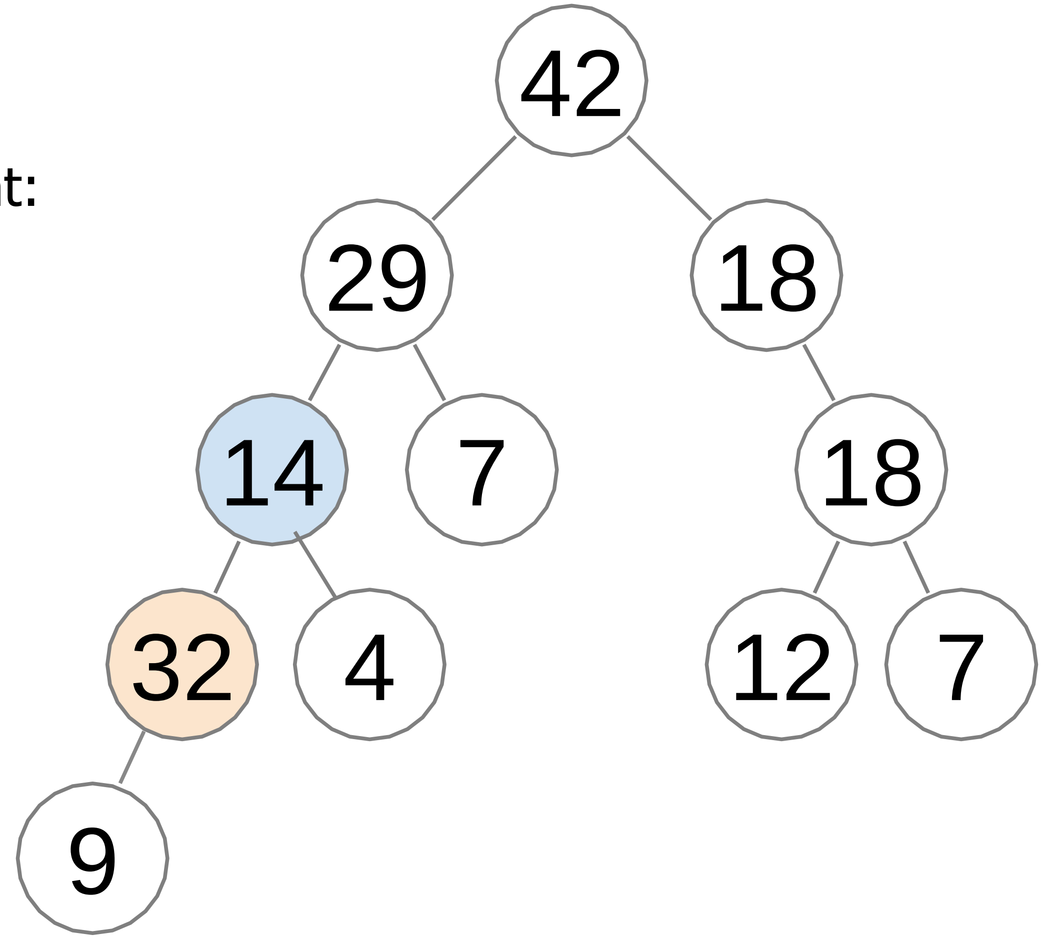
Heap operations: ***`sift_up(e)`***

if current element is
bigger than the parent:
swap



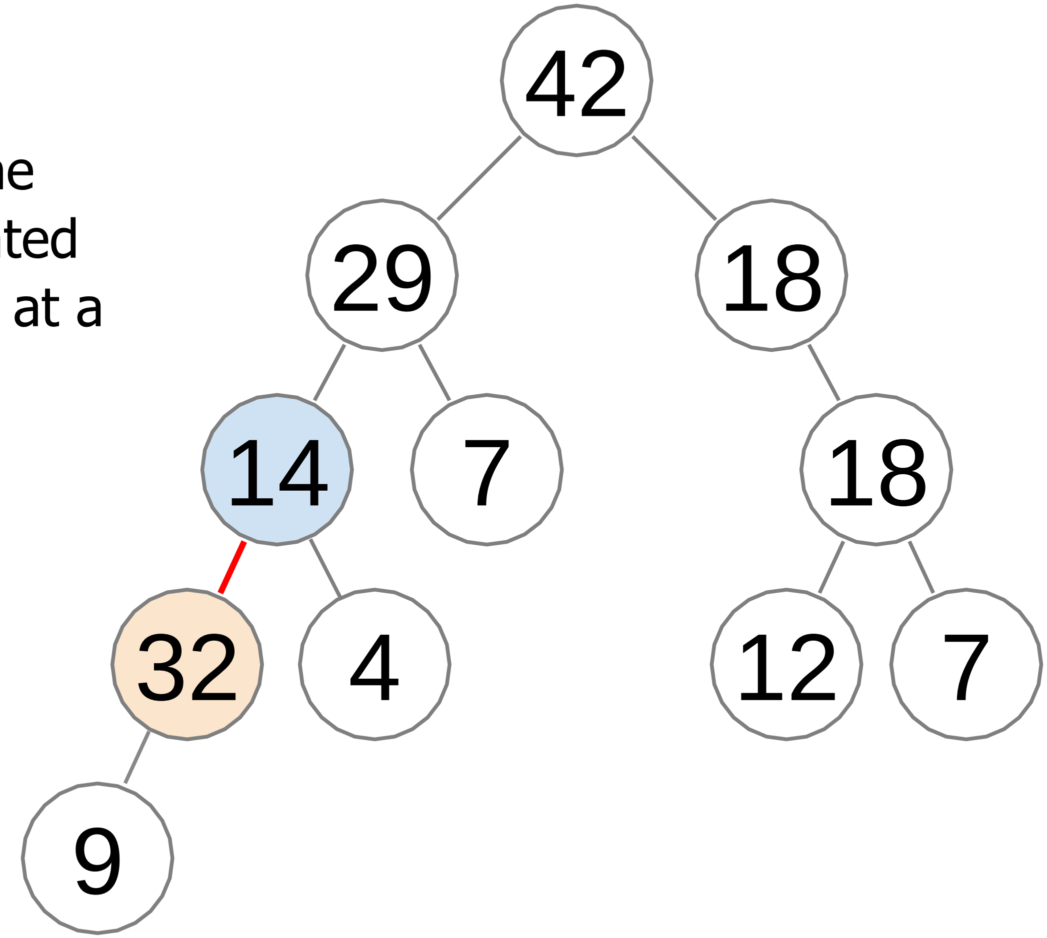
Heap operations: `sift_up(e)`

if current element is
bigger than the parent:
swap



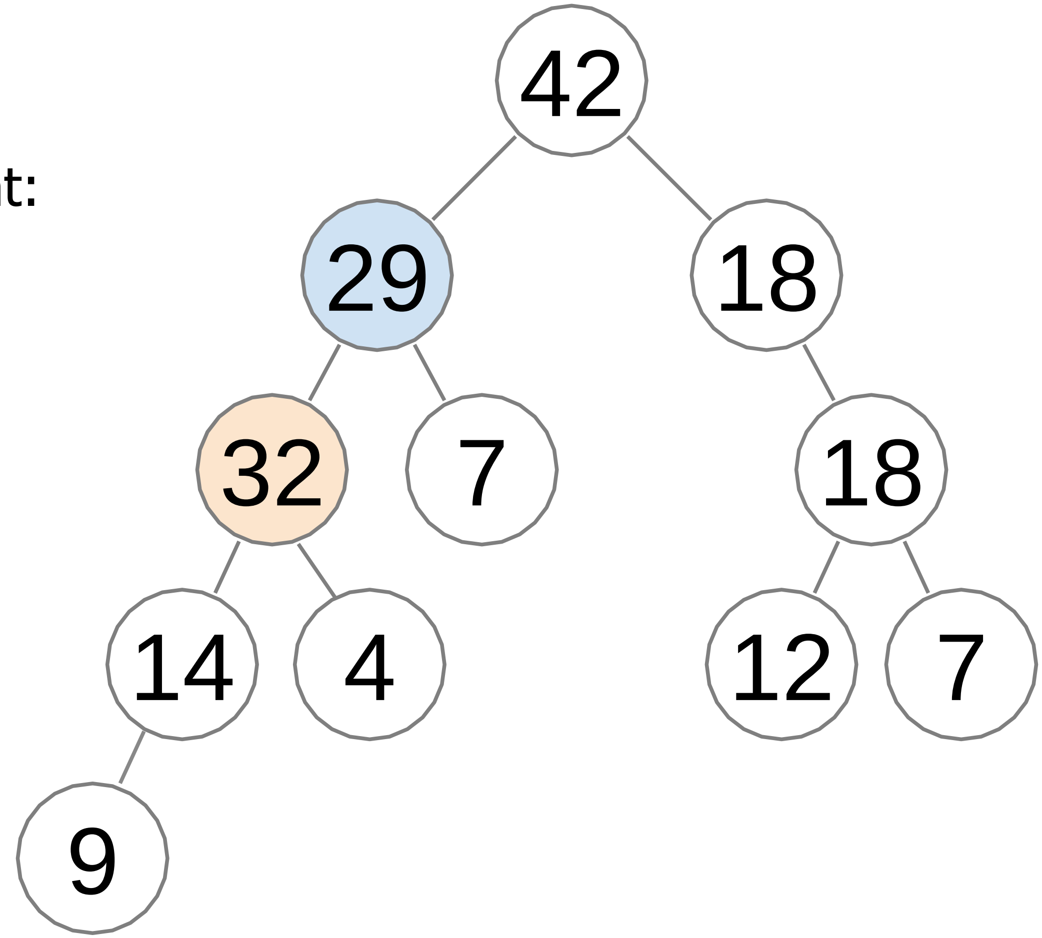
Heap operations: `sift_up(e)`

this works because the heap property is violated only on a single edge at a time



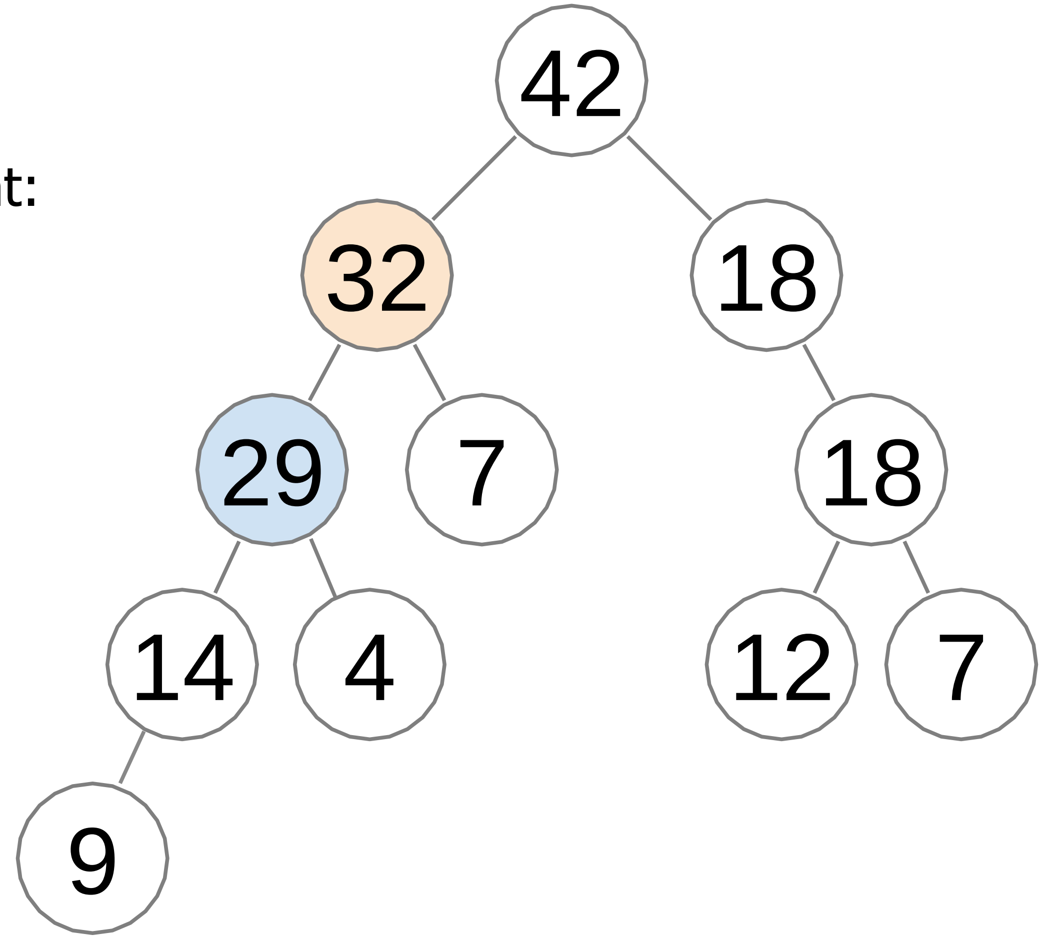
Heap operations: `sift_up(e)`

if current element is
bigger than the parent:
swap



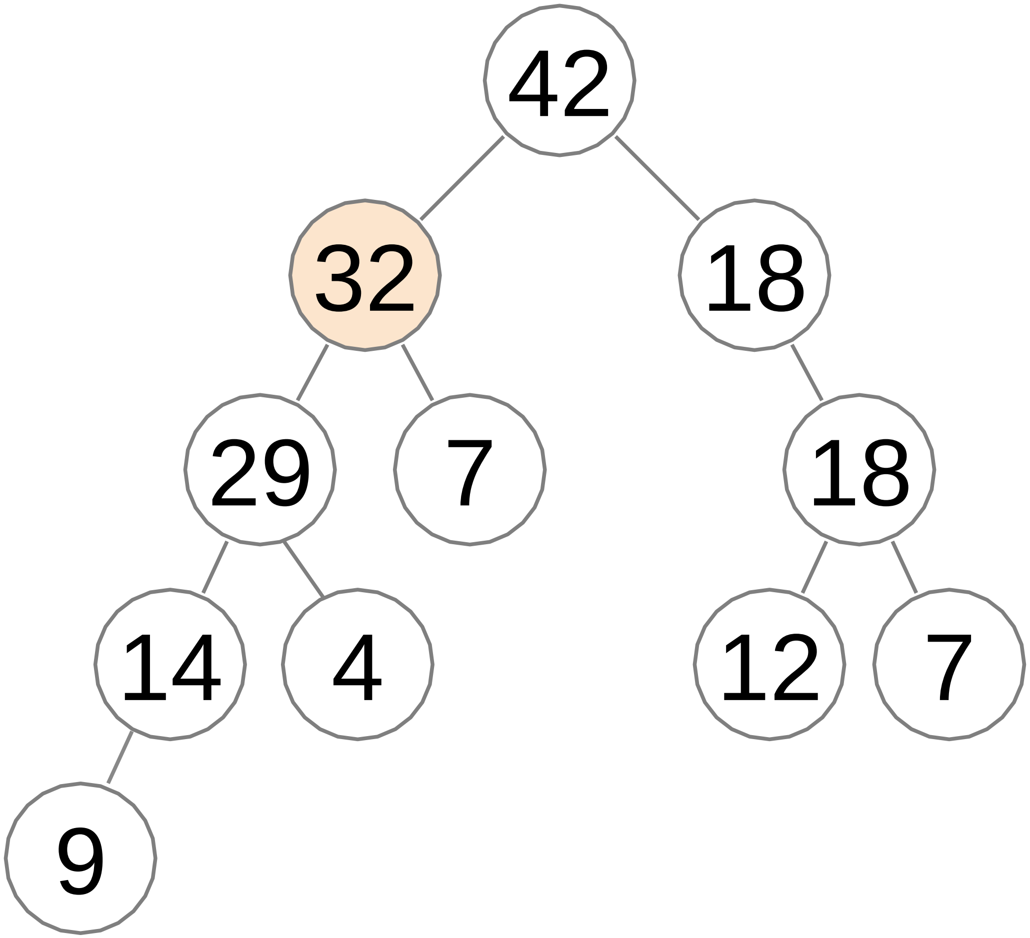
Heap operations: ***`sift_up(e)`***

if current element is
bigger than the parent:
swap



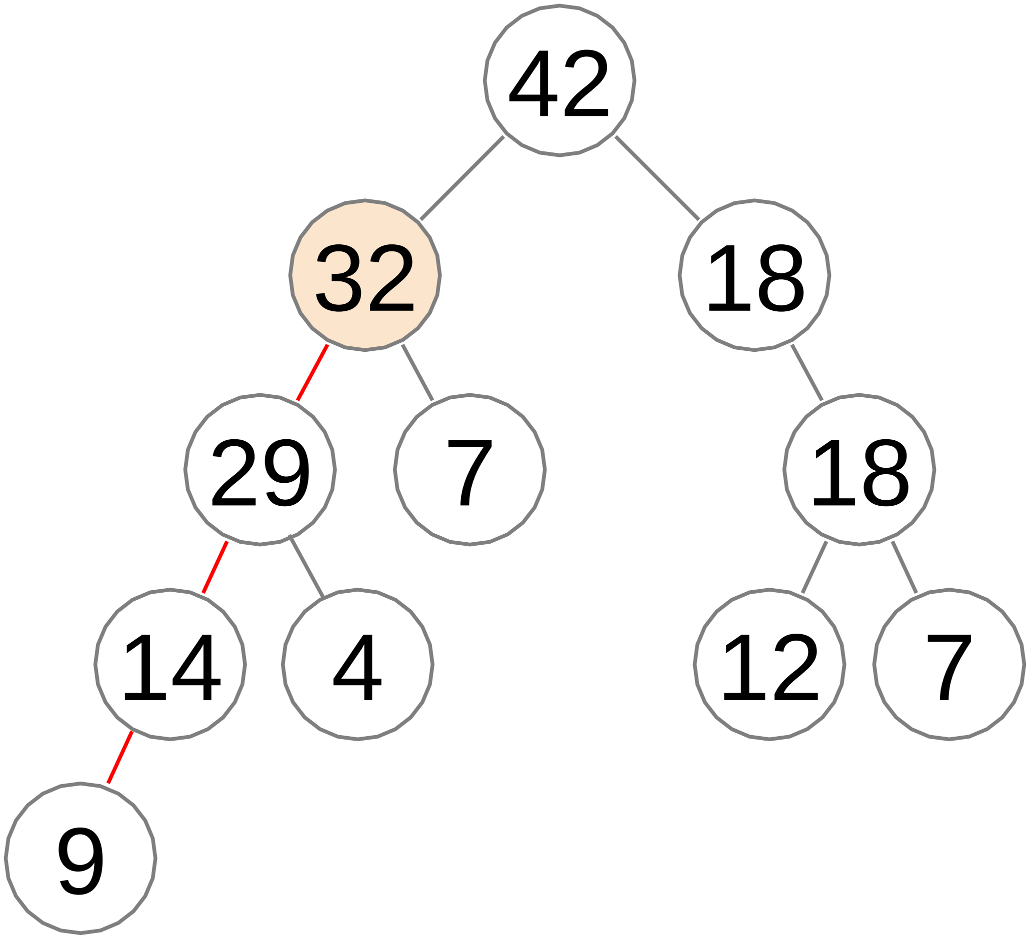
Heap operations: `sift_up(e)`

heap property is restored



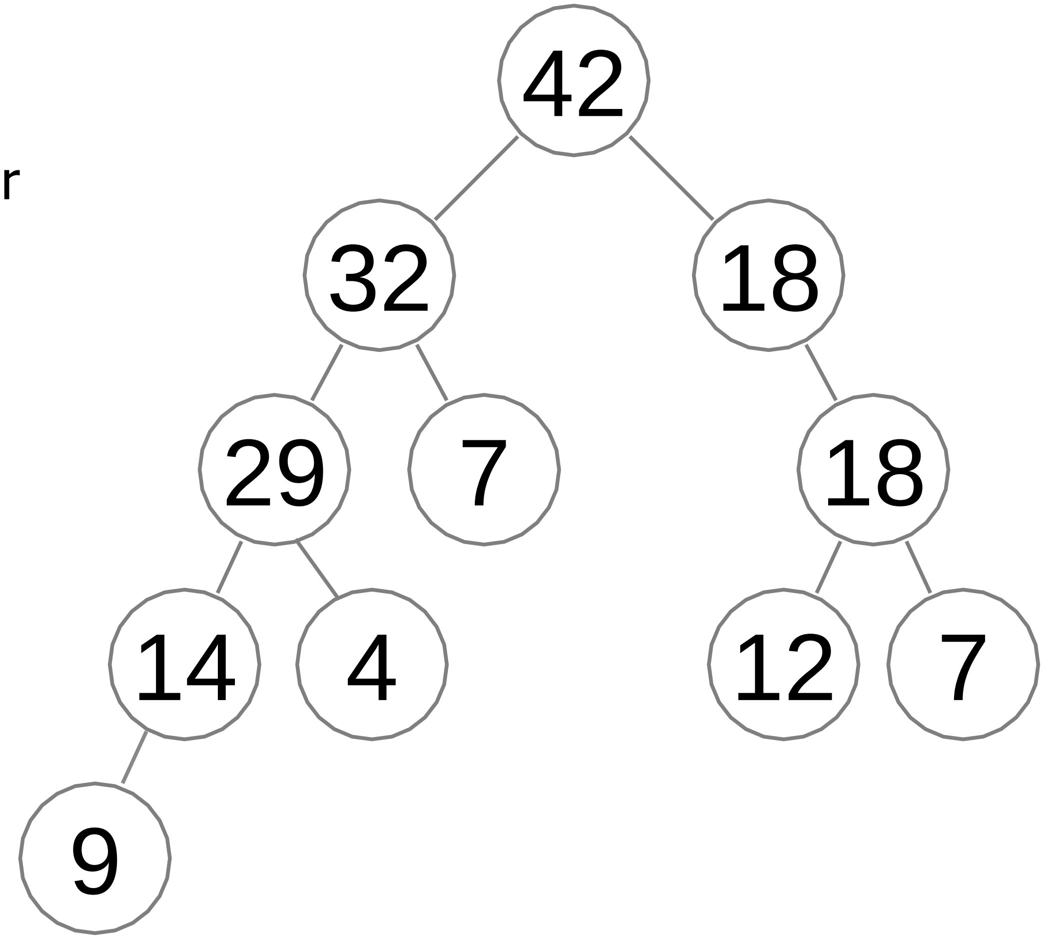
Heap operations: `enqueue(e)`

running time of *enqueue* depends on how many times we need to *swap*



Heap operations: `enqueue(e)`

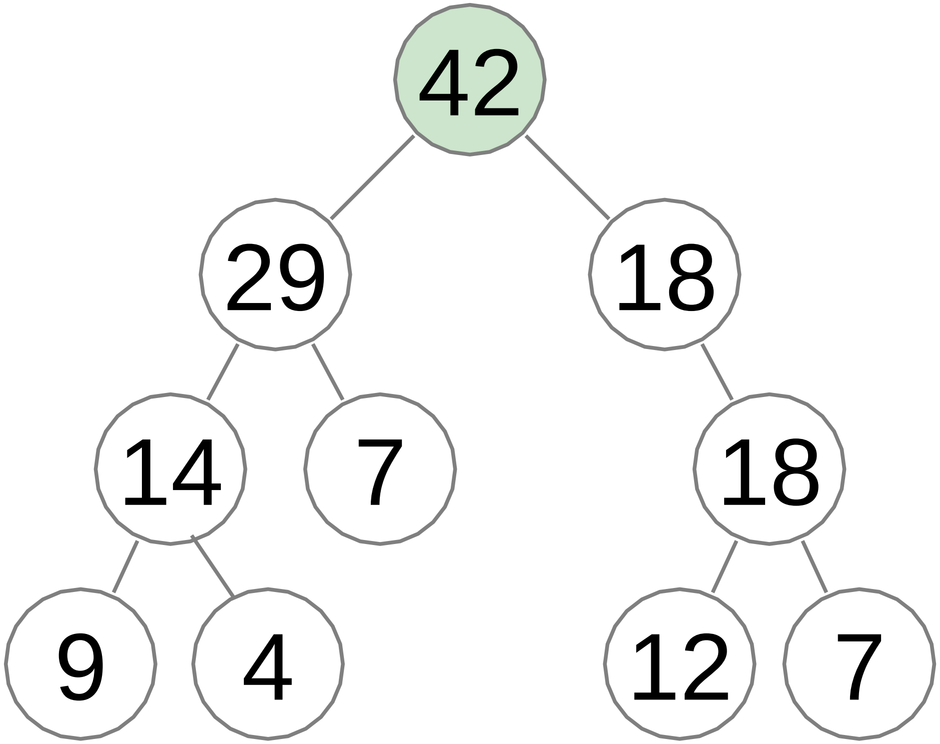
with each swap, the problematic node moves one node closer to the root



running time: O(tree height)

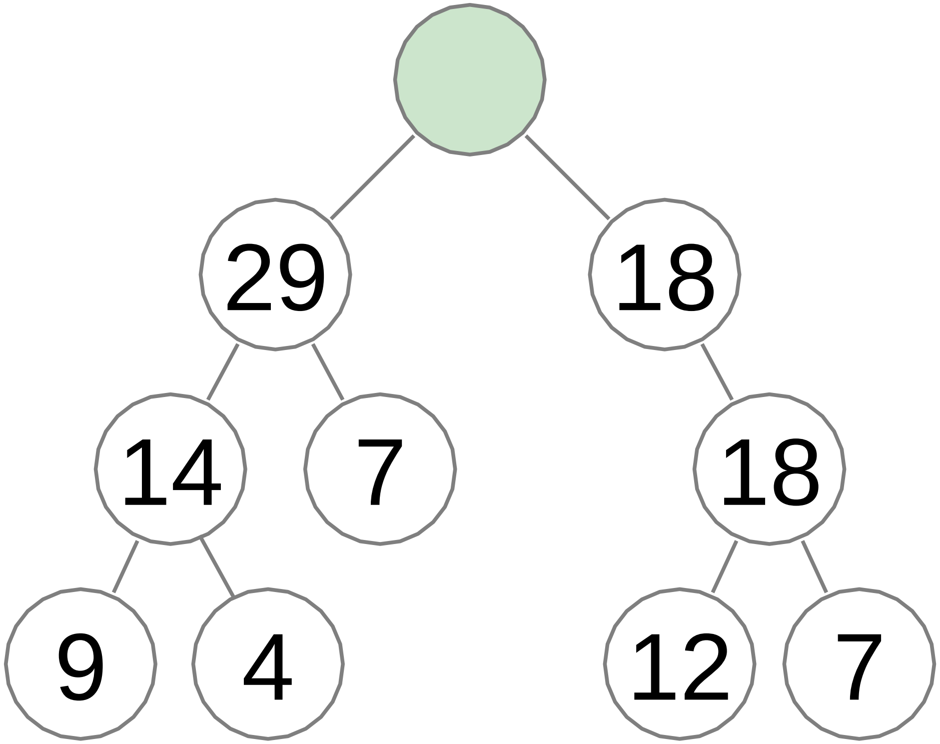
Heap operations: `dequeue`

remove and return the root value



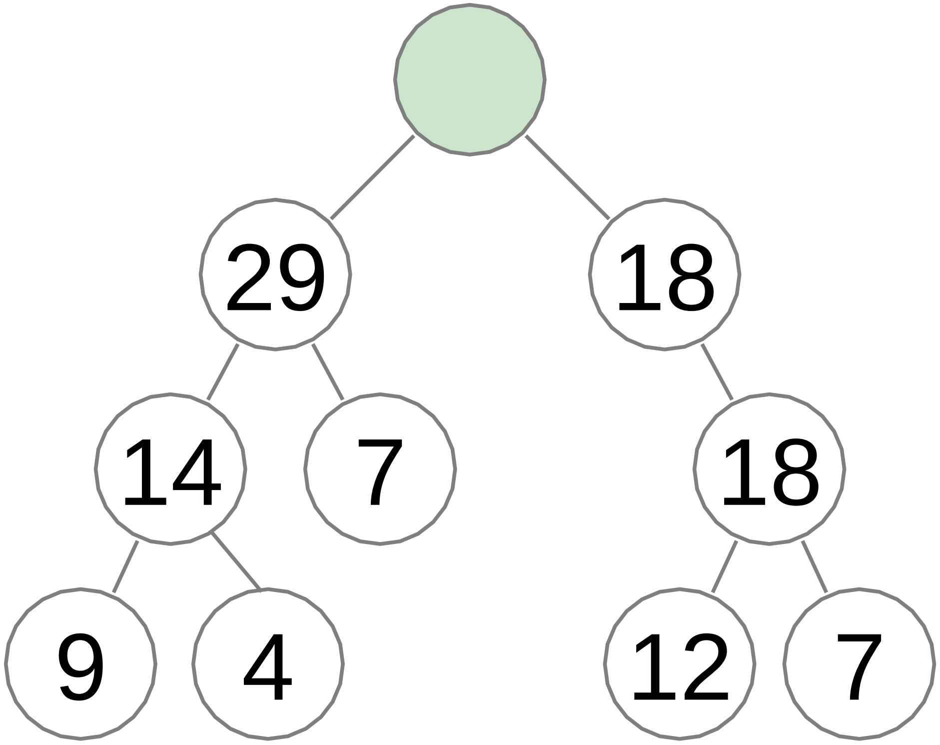
Heap operations: **`dequeue`**

remove the root value



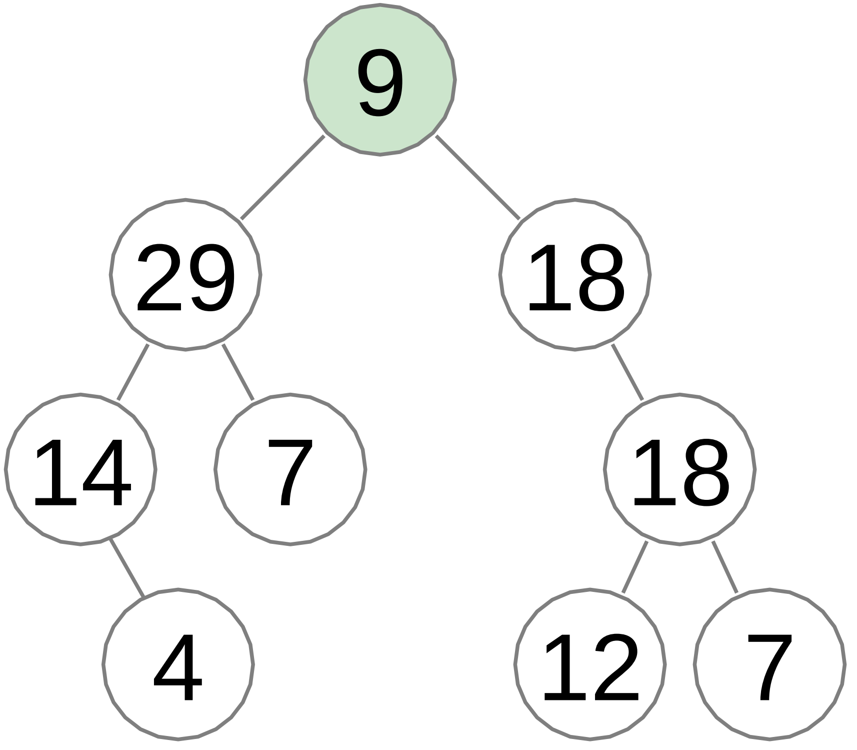
Heap operations: ***dequeue***

replace the empty node value with any leaf node value and remove the leaf



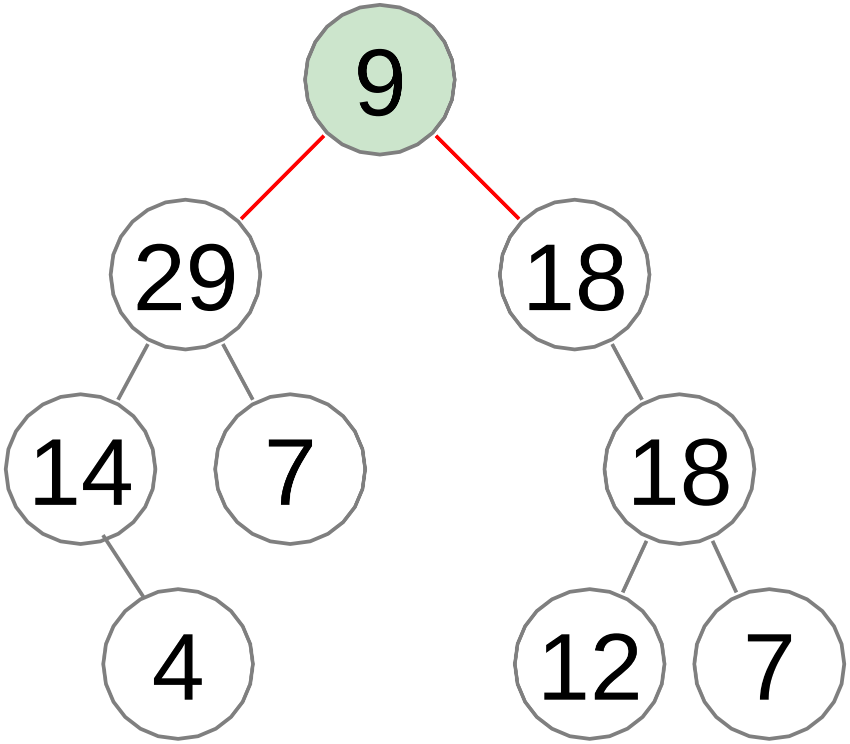
Heap operations: ***dequeue***

replace the empty node value with any leaf node value and remove the leaf



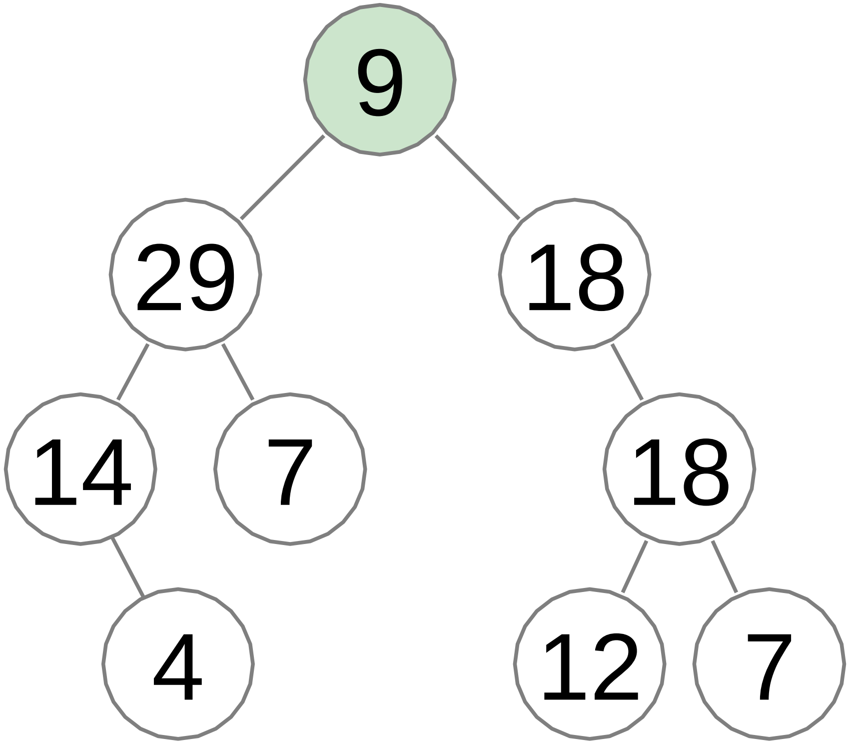
Heap operations: ***dequeue***

again, this may violate the heap property



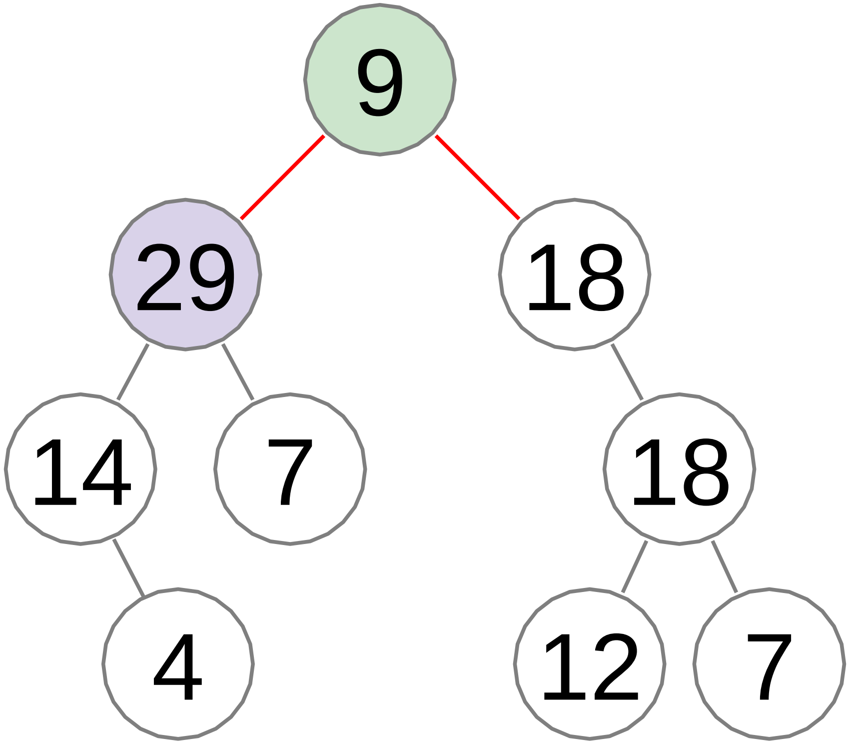
Heap operations: ***dequeue***

to fix it we let the problematic node *sift down*



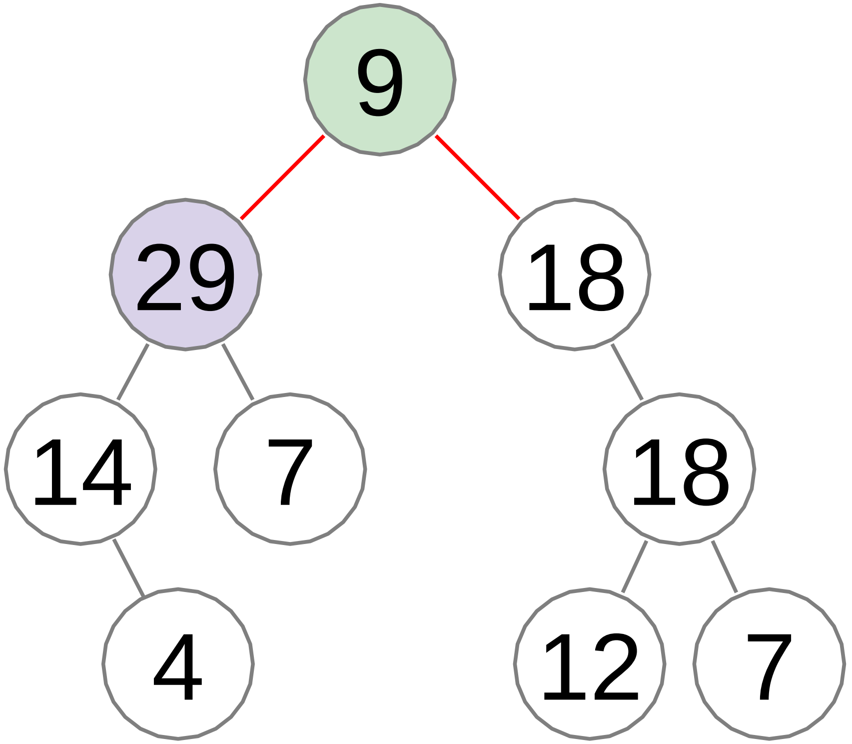
Heap operations: ***sift_down(e)***

if current node is smaller than one of its children, swap it with the largest child



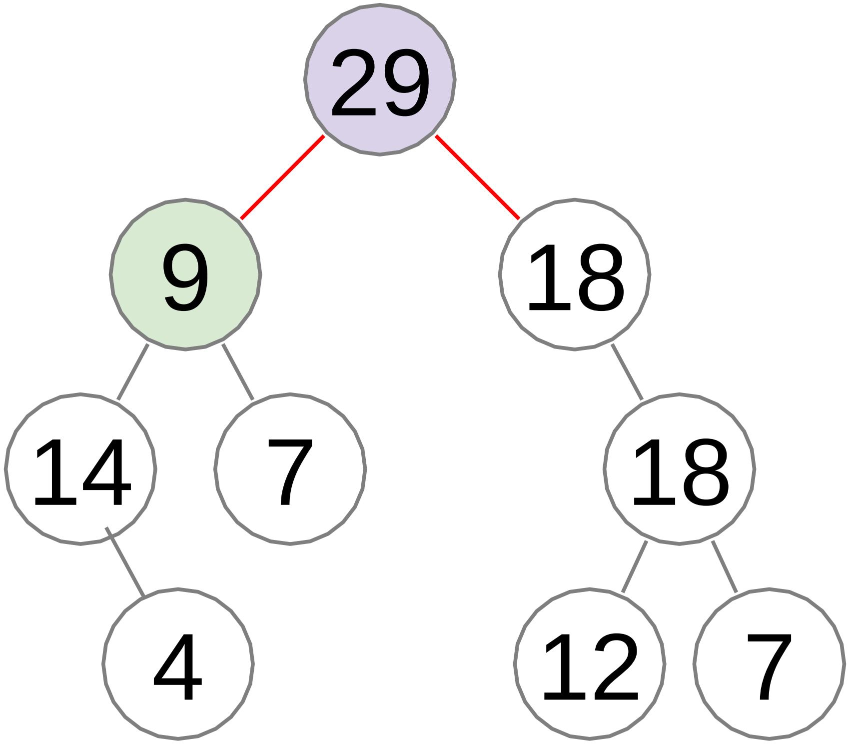
Heap operations: ***sift_down(e)***

swapping with the largest child automatically restores both broken edges



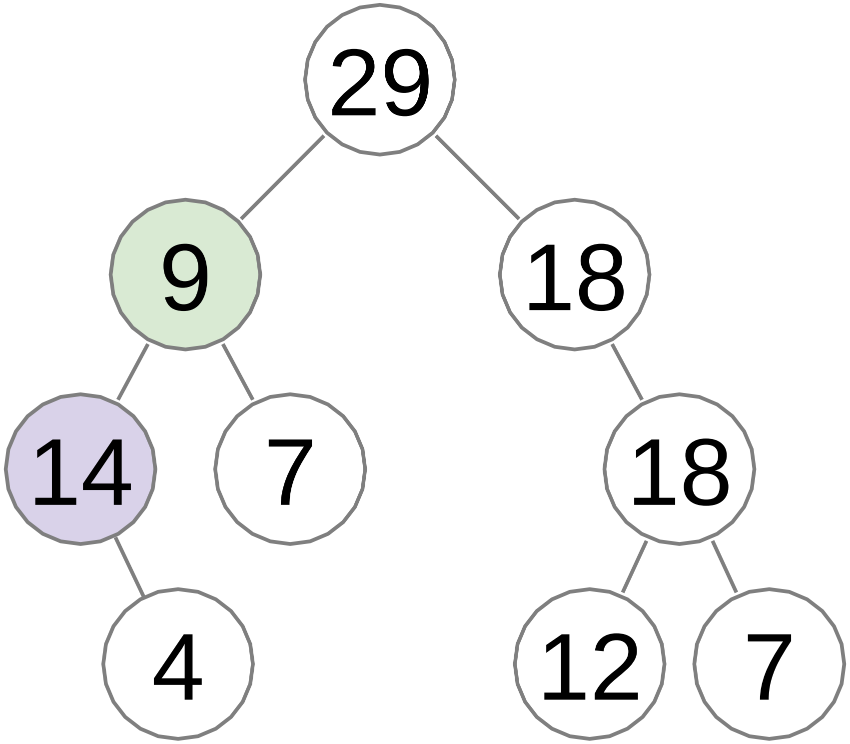
Heap operations: `sift_down(e)`

swapping with the largest child automatically restores both broken edges



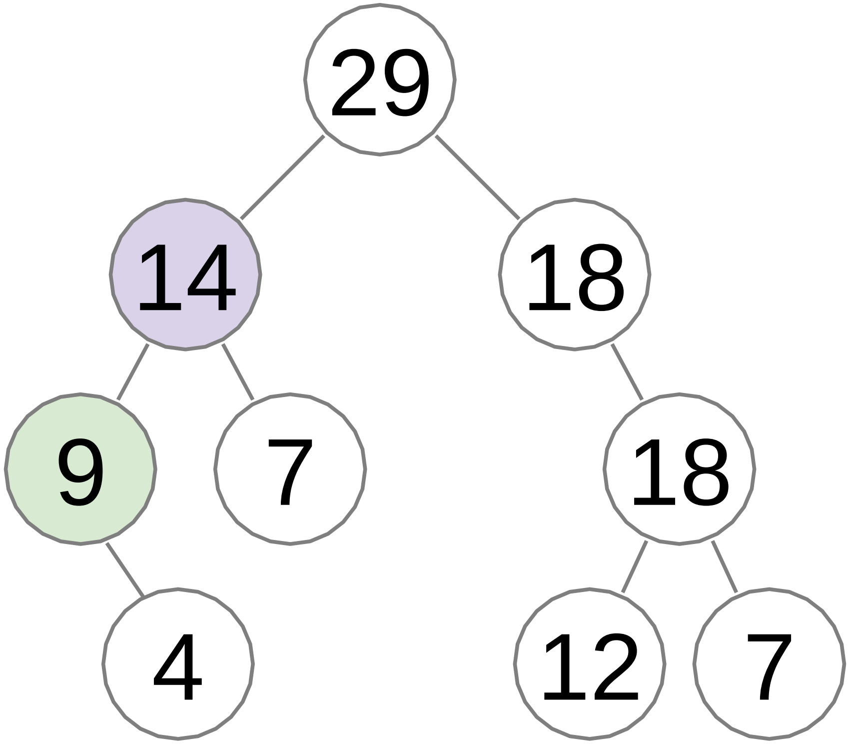
Heap operations: ***`sift_down(e)`***

if current node is smaller than one of its children, swap it with the largest child



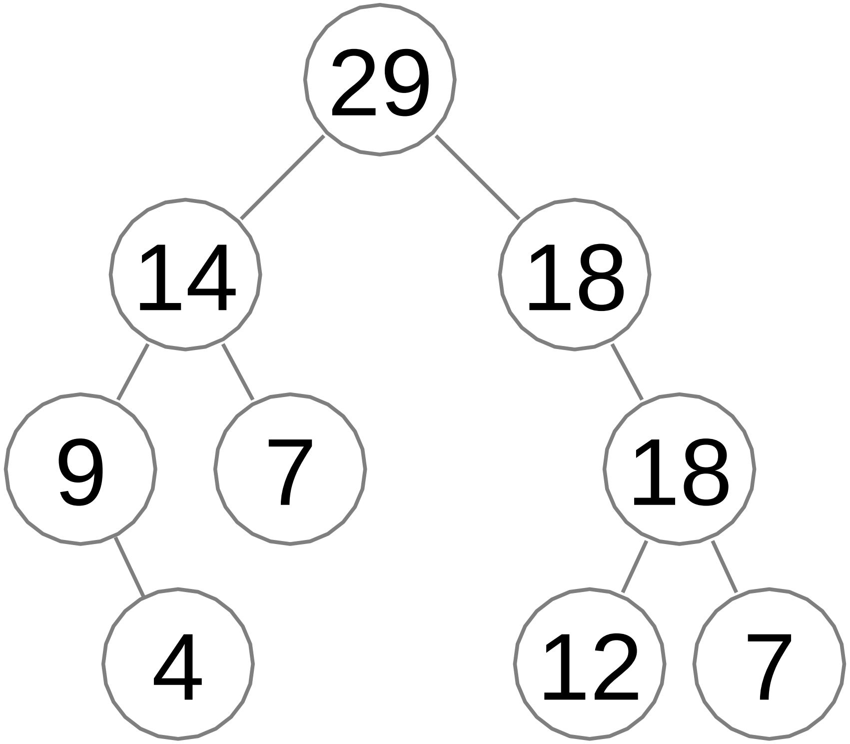
Heap operations: `sift_down(e)`

if current node is smaller than one of its children, swap it with the largest child



Heap operations: `sift_down(e)`

the heap property is restored



Suppose you have a Binary Search Tree. Is it also a heap?

A. Yes

B. No

C. Sometimes

Suppose you have a binary heap. Is it also a binary search tree?

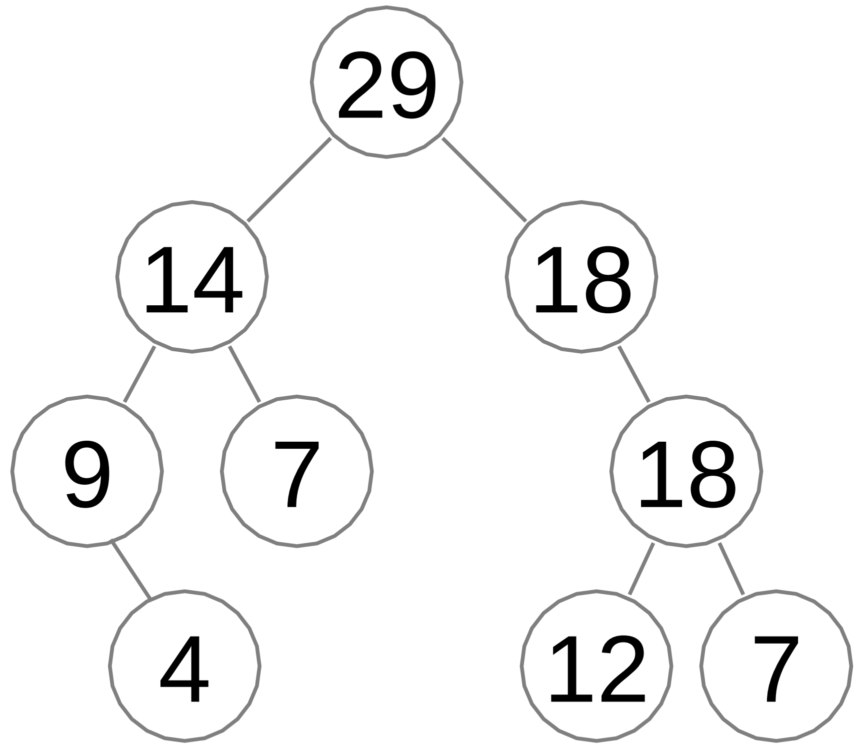
A. Yes

B. No

C. Sometimes

Heap operations: enqueue and dequeue

Running time depends on how many times the *swap* is performed to restore the heap



running time: O(tree height)

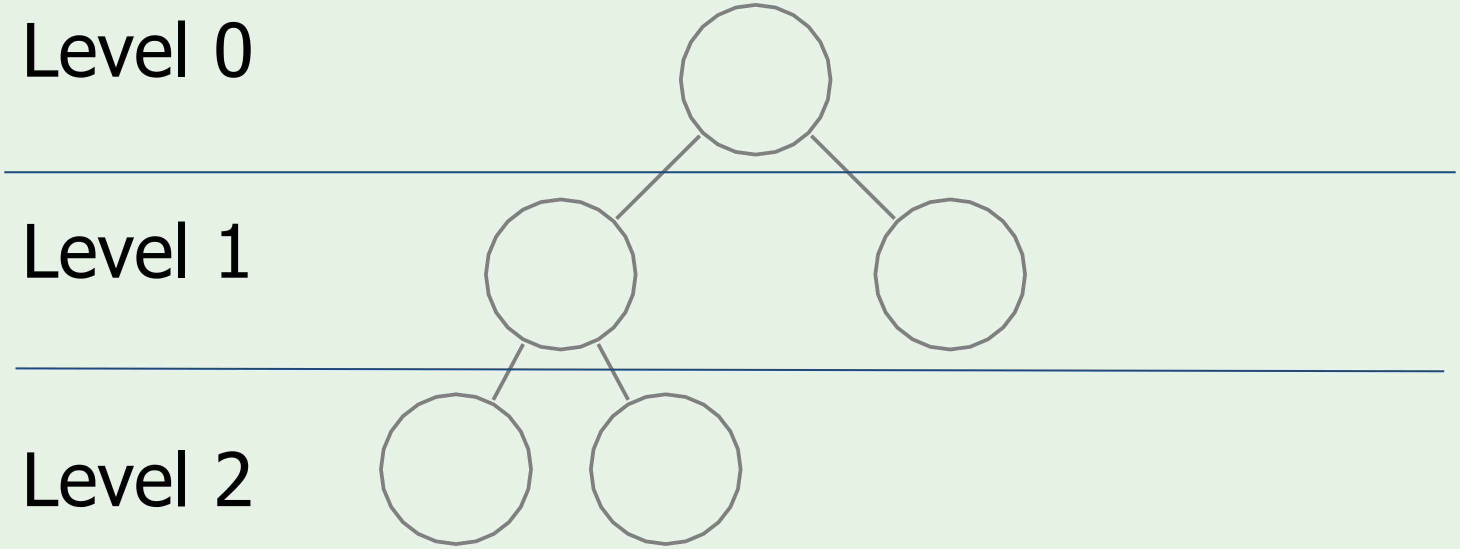
We want a tree with the min possible height

How to Keep a Tree Shallow?

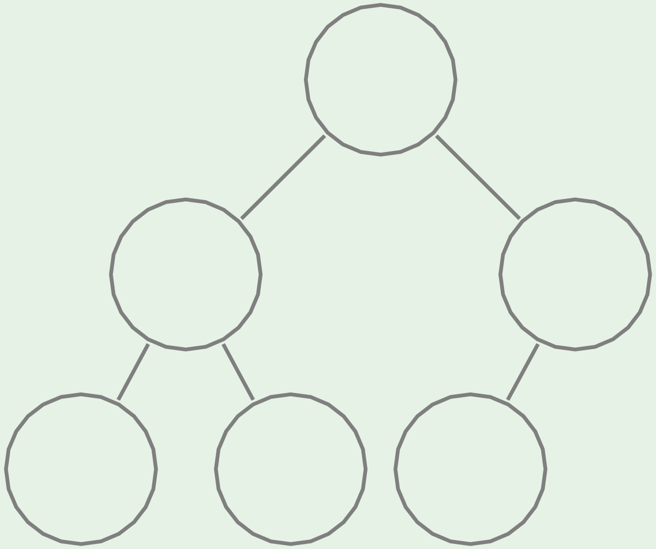
Definition

A binary tree is ***complete*** if all its levels are at full capacity except possibly the last one which is filled from left to right.

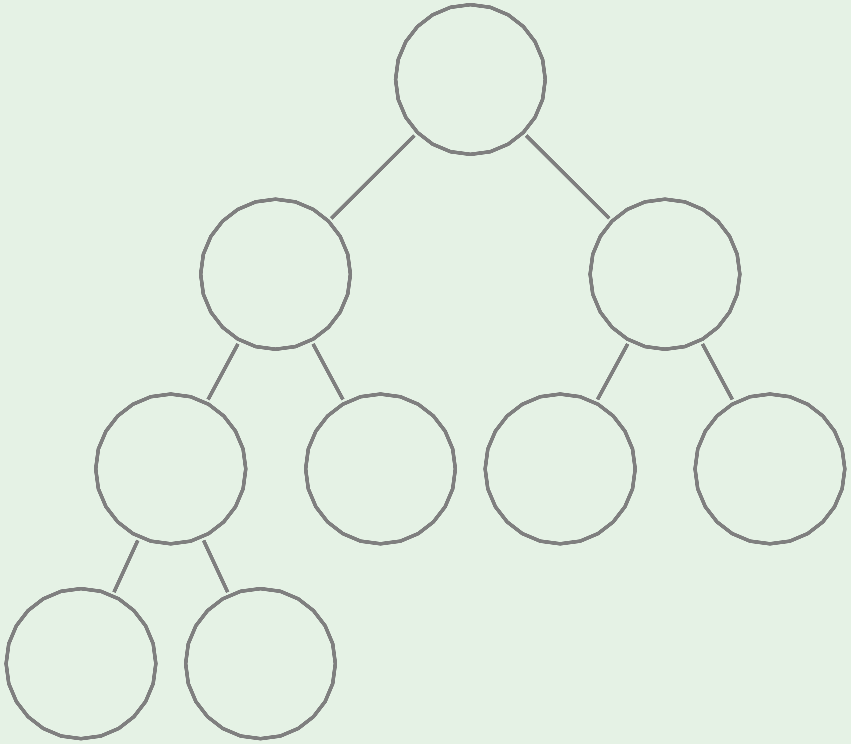
Example: complete binary tree



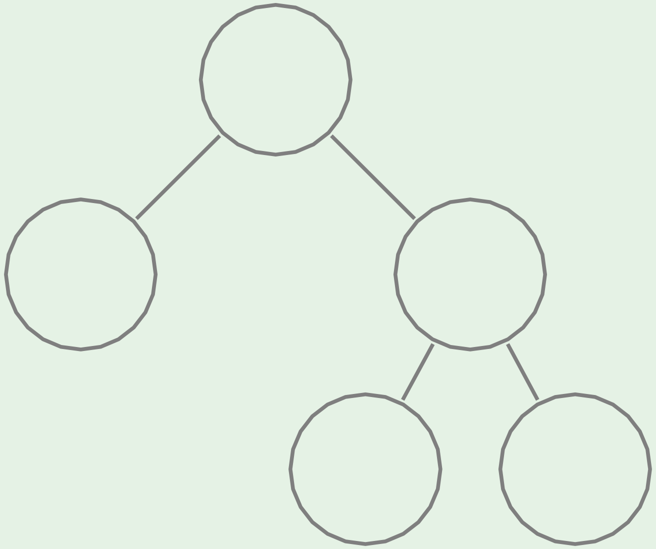
Complete binary tree ?



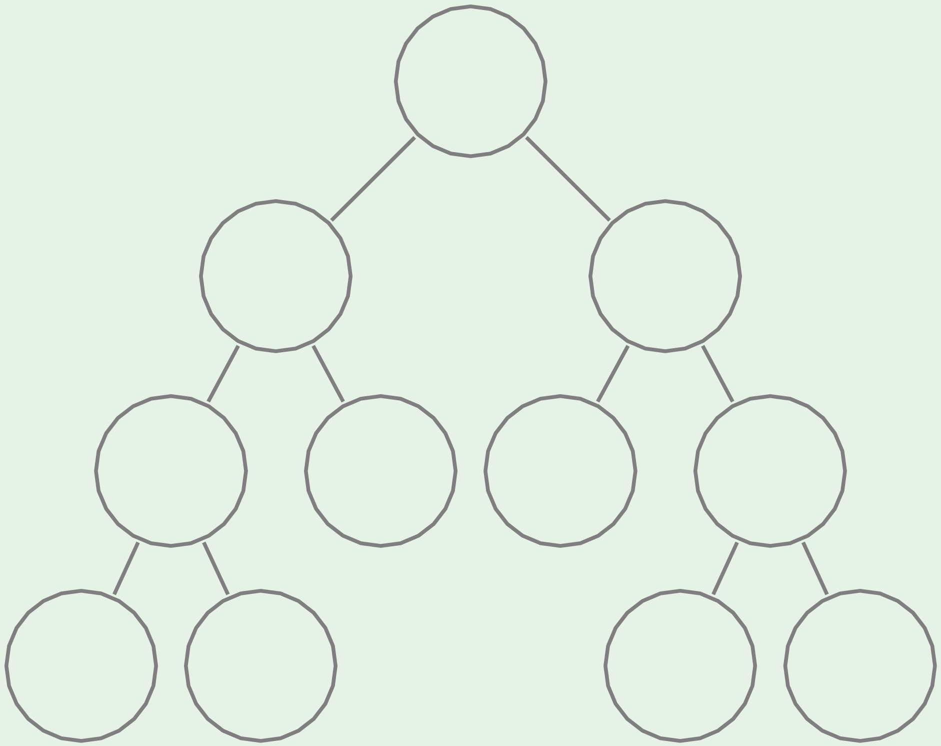
Complete binary tree ?



Complete binary tree ?



Complete binary tree ?



Advantage of Complete Binary Trees: low height

Theorem

A complete binary tree with n total nodes has height at most $O(\log n)$.

Proof

- Complete the last level of the tree if it is not full to get a **full** binary tree.
- This full tree has $n' \geq n$ nodes and the same height h.
- At level 0 we have $2^0=1$ node, at the first level: $2^1=2$ nodes, at level k: 2^k nodes, and the total number of levels is h-1. Then the total number of nodes:

$$n' = 1 + 2^1 + 2^2 + \ldots 2^{h-1} = \frac{2^{(h-1)+1} - 1}{2-1} = 2^h - 1$$

(sum of geom. series)

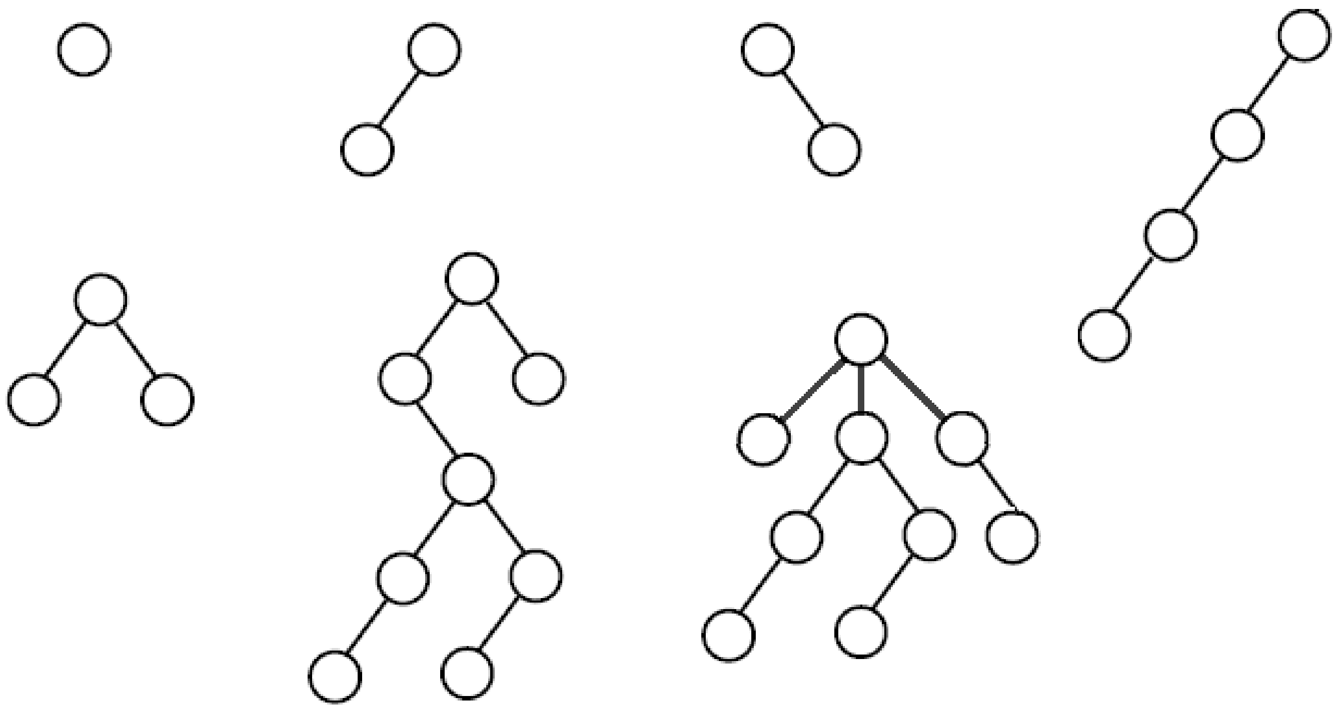
- Note that $n' \leq 2n$, because the actual total number of nodes n is between $2^{h-2+1} - 1 + 1 = 2^{h-1}$ and $2^h - 1$
- Then $n' = 2^h - 1$ and hence:
  $h = \log_2(n' + 1) \leq \log_2(2n + 1) = O(\log n)$. ■

If we store Heap as a Complete Binary Tree:

- → *Top* in time O(1)
- → *Dequeue* in time O(log *n*)
- → *Enqueue* in time O(log *n*)

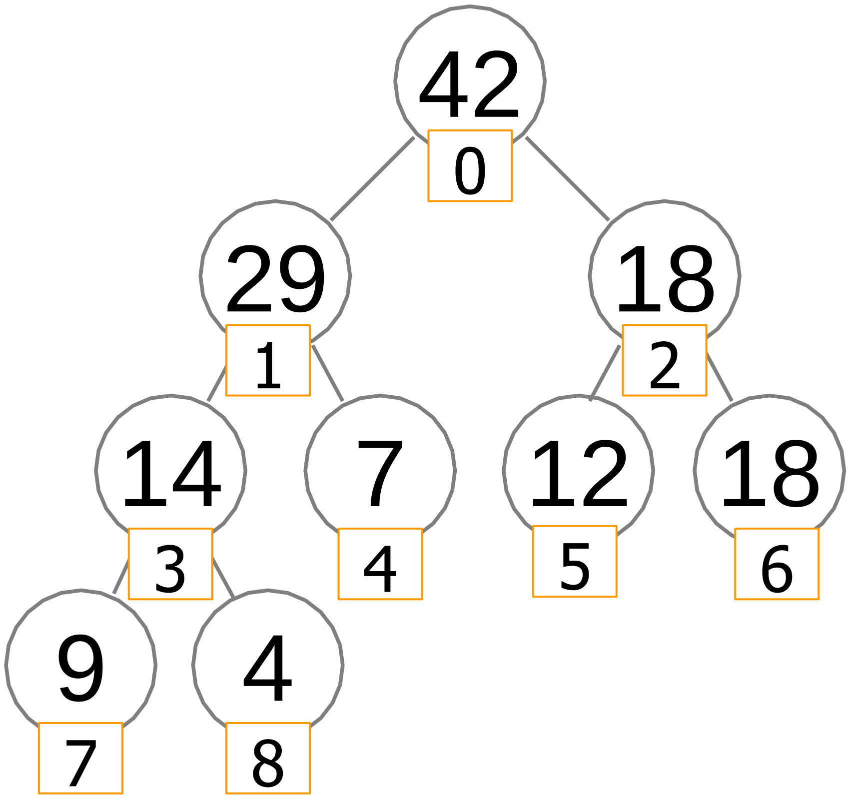
As long as we keep the tree complete

How many of these structures represent a complete binary tree?

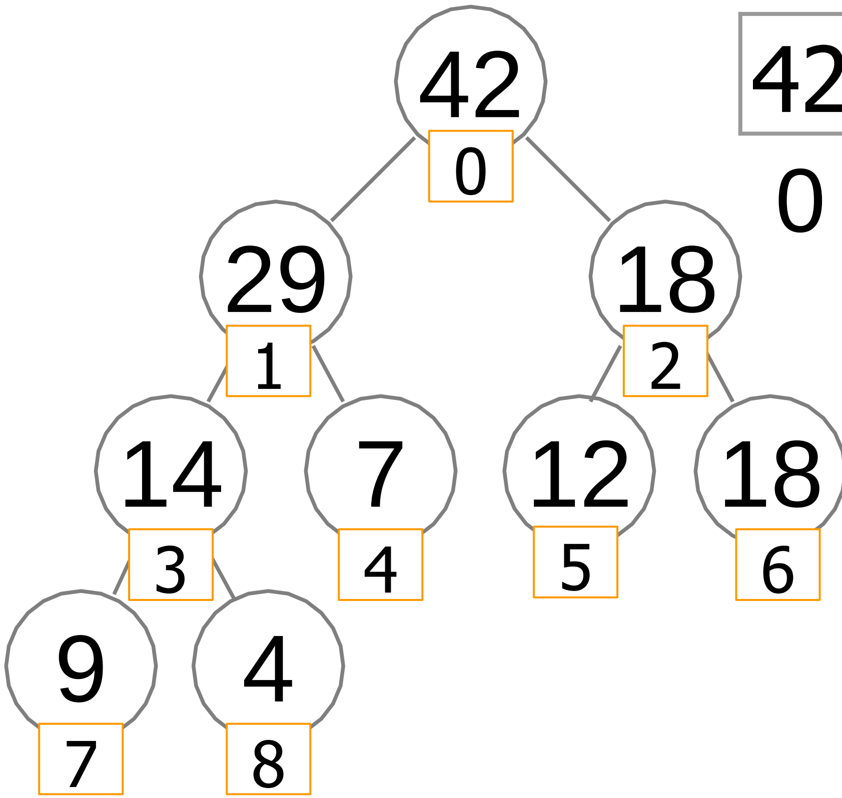


A. 0-1
B. 2
C. 3
D. 4
E. 5-8

A Complete Binary Tree can be stored in an Array

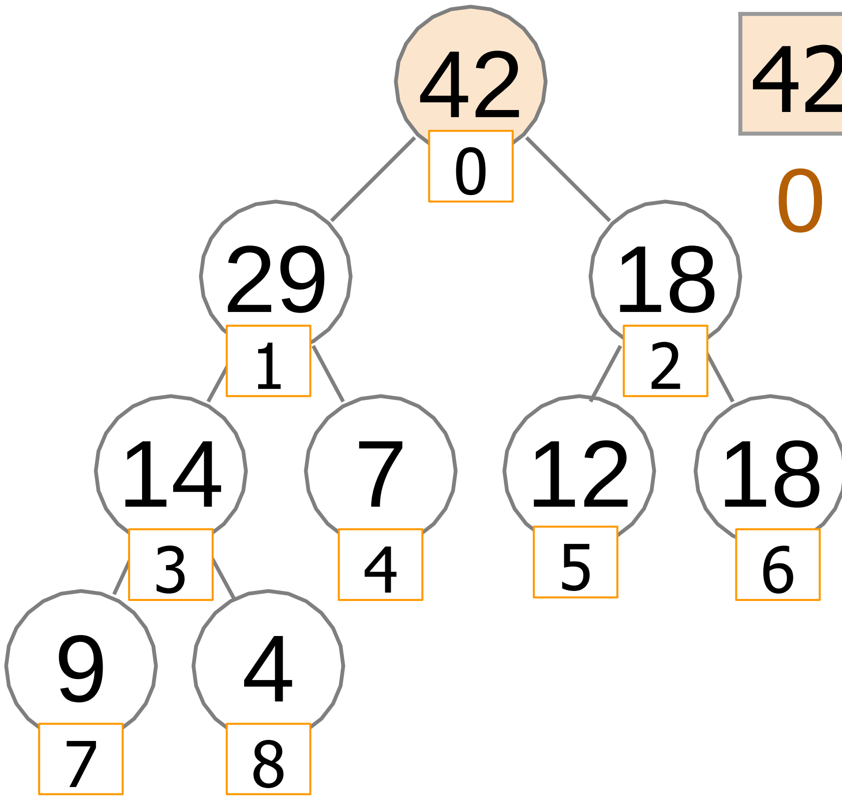


A Complete Binary Tree can be stored in an Array



42	29	18	14	7	12	18	9	4
0	1	2	3	4	5	6	7	8

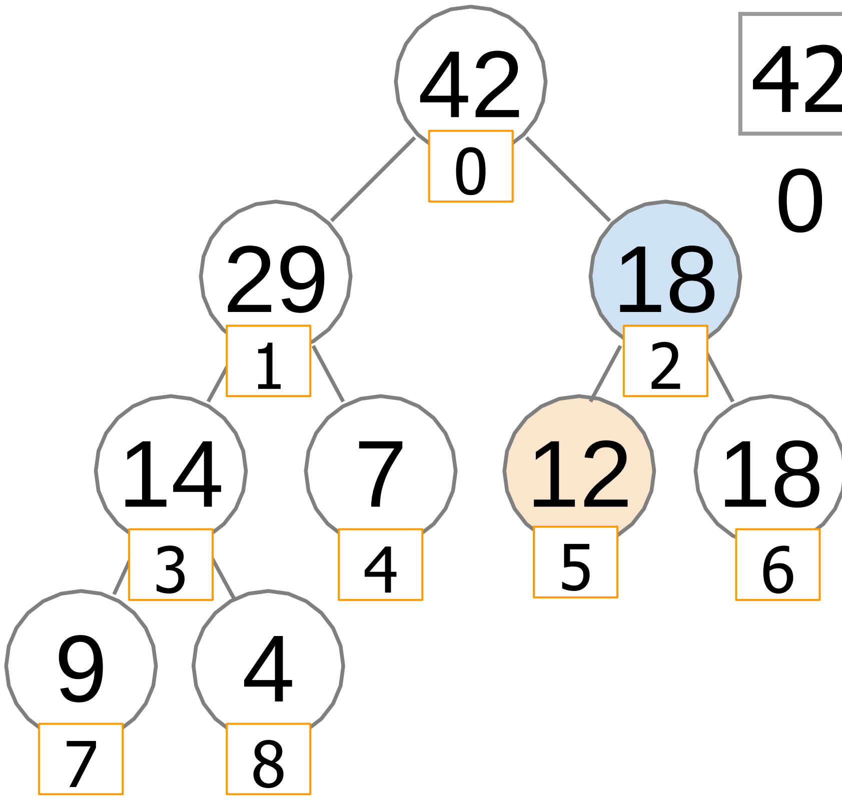
A Complete Binary Tree can be stored in an Array



0	1	2	3	4	5	6	7	8
42	29	18	14	7	12	18	9	4

top: A[0]

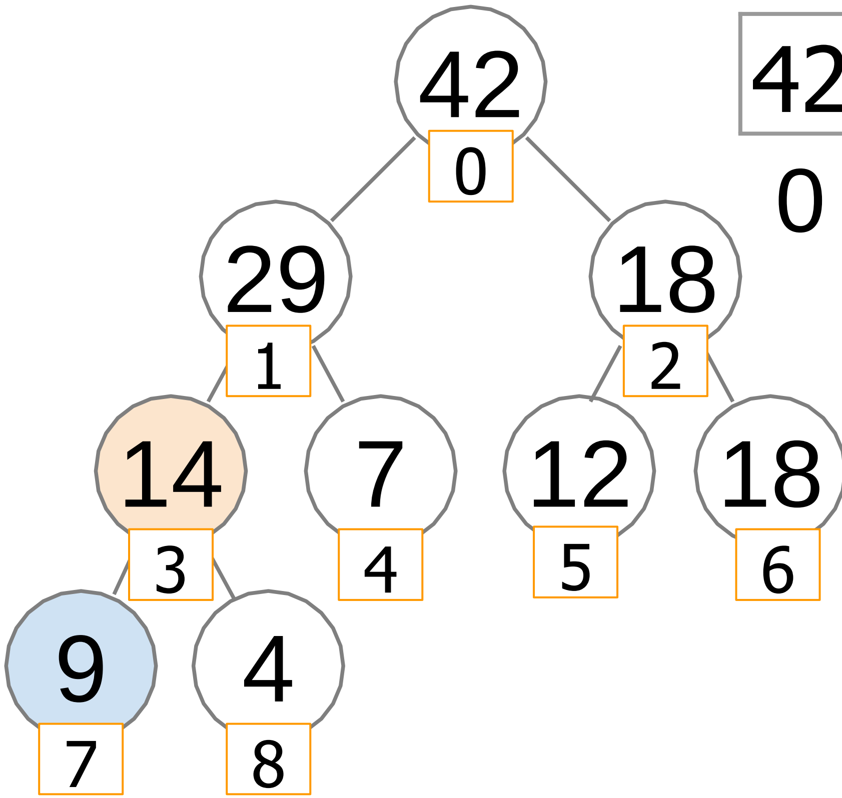
Tree operations in a heap array



42	29	18	14	7	12	18	9	4
0	1	2	3	4	5	6	7	8

$$\text{parent}(A[i]) = A[\lfloor (i-1)/2 \rfloor]$$

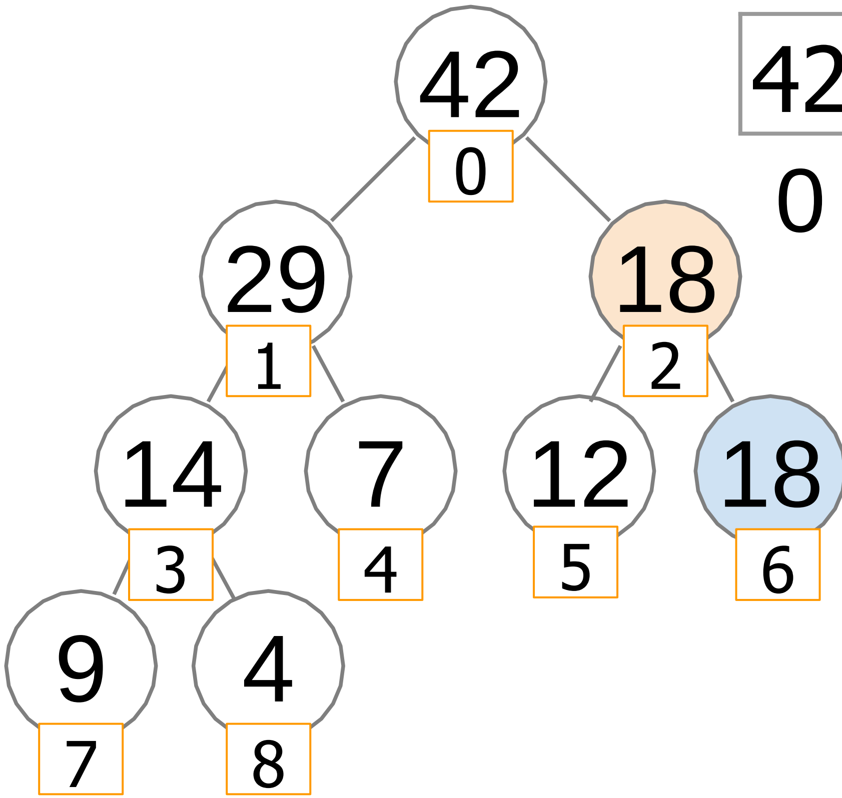
Tree operations in a heap array



42	29	18	14	7	12	18	9	4
0	1	2	3	4	5	6	7	8

$$\text{left\_child}(A[i]) = A[2i + 1]$$

Tree operations in a heap array

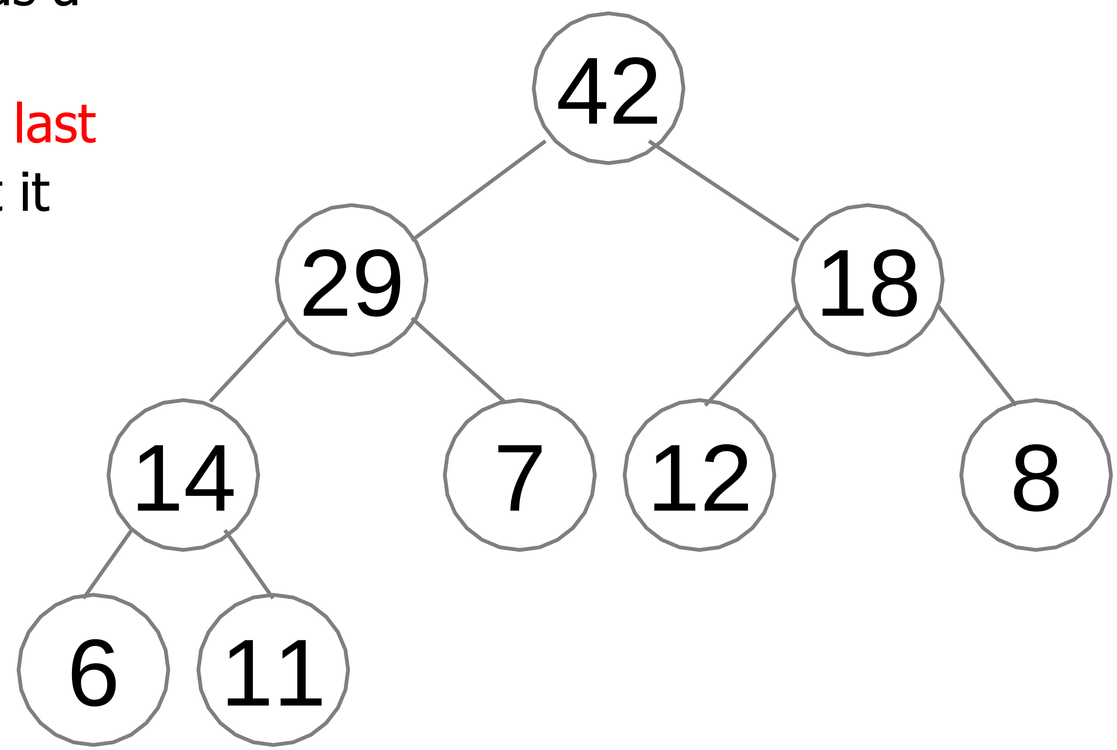


42	29	18	14	7	12	18	9	4
0	1	2	3	4	5	6	7	8

right_child(A[i]) = A[$2i + 2$]

Heap array: ***`enqueue(33)`***

to add an element, insert it as a leaf in the rightmost vacant position in the last level (the last position of the array) and let it *sift up*

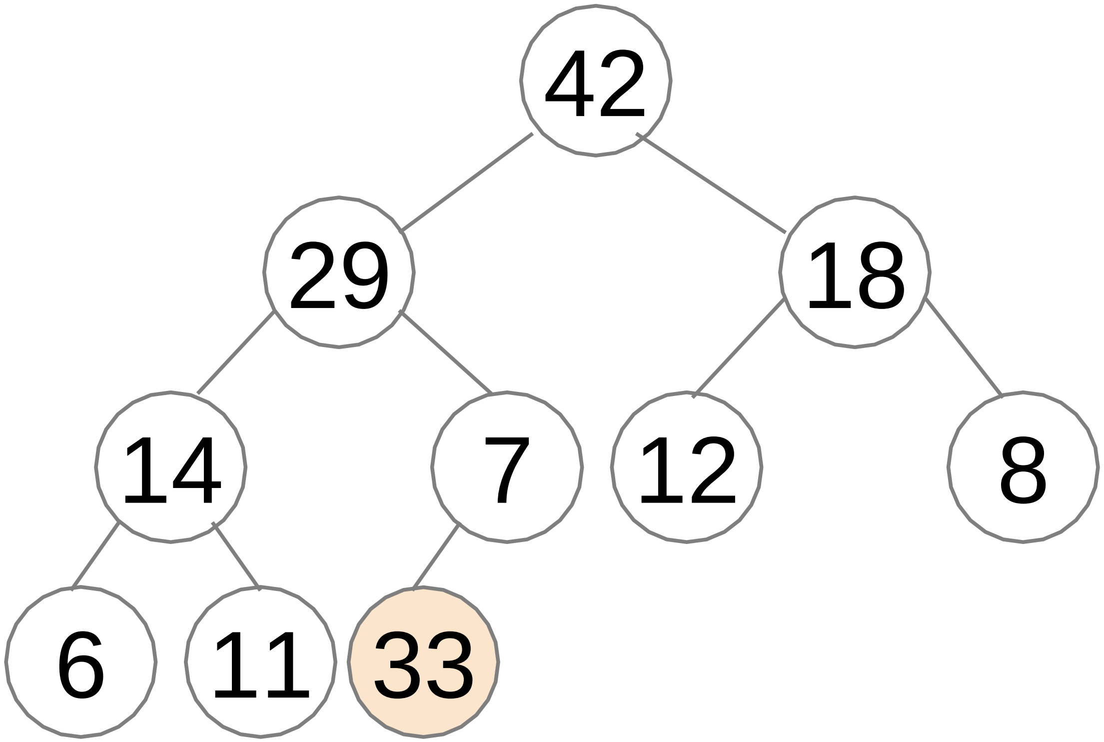


0	1	2	3	4	5	6	7	8	9
42	29	18	14	7	12	8	6	11	

Heap array: ***enqueue(33)***

$$\text{parent}(i) = \lfloor (i-1)/2 \rfloor$$

parent(9) = 4
swap(A[9],A[4])



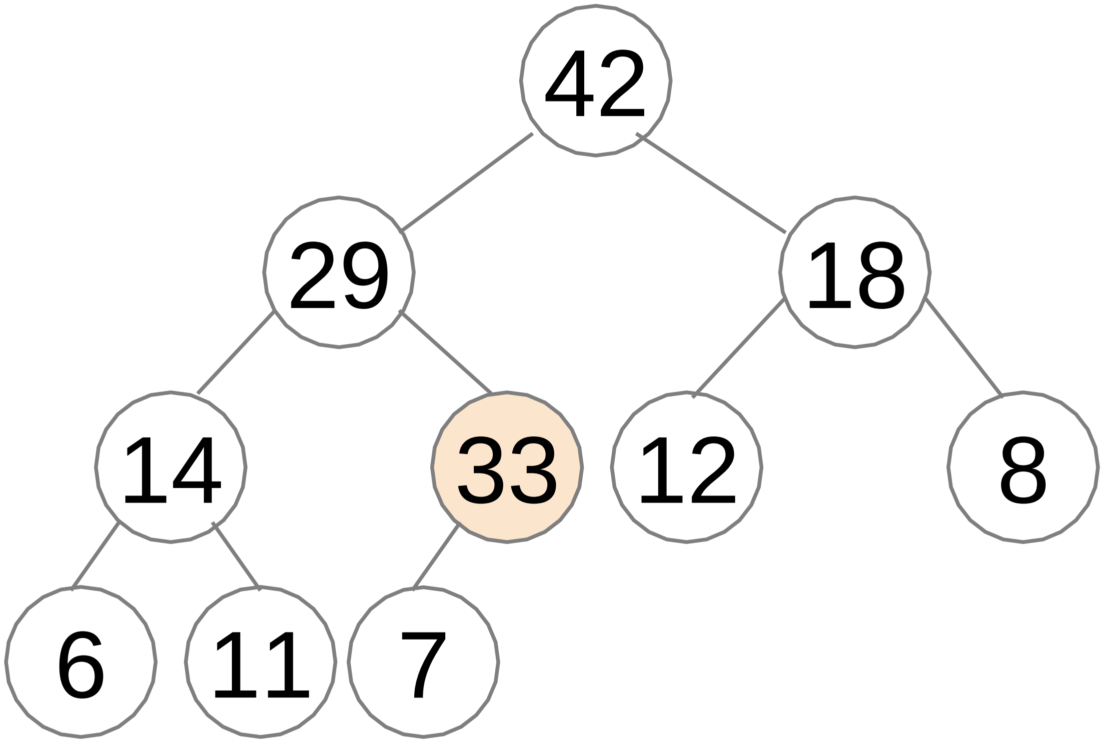
42	29	18	14	7	12	8	6	11	33
0	1	2	3	4	5	6	7	8	9

Heap array: *enqueue(33)*

parent(i) = $\lfloor (i-1)/2 \rfloor$

parent(9) = 4
swap(A[9],A[4])

parent(4) = 1
swap(A[4],A[1])



42	29	18	14	33	12	8	6	11	7
0	1	2	3	4	5	6	7	8	9

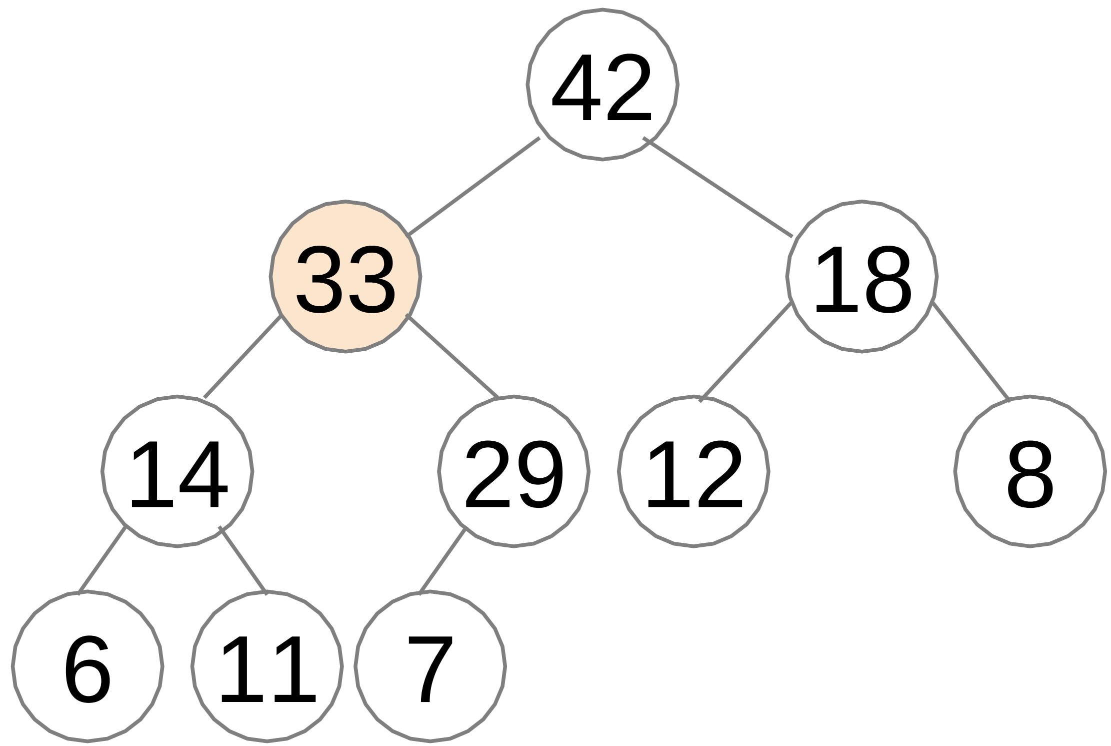
Heap array: *enqueue(33)*

$$\text{parent}(i) = \lfloor (i-1)/2 \rfloor$$

parent(9) = 4
swap(A[9],A[4])

parent(4) = 1
swap(A[4],A[1])

parent(1) = 0 OK
stop



42	33	18	14	29	12	8	6	11	7
0	1	2	3	4	5	6	7	8	9

Heap array: `dequeue()`

Similarly, to extract the maximum value, replace the root by the last leaf and let it *sift down*

Binary **Min**-Heap

Definition

Binary **min**-heap is a binary tree where the value of each node is **at most** the values of its children.

Can be implemented similarly to max-heap

How many swaps will we do after we call dequeue() on this **min**-heap?

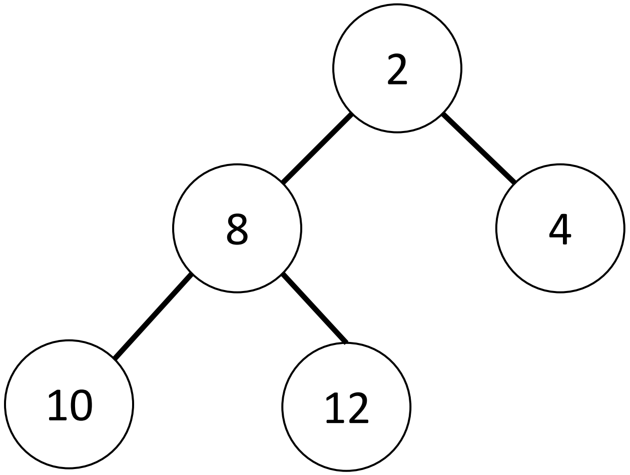
A. 0

B. 1

C. 2

D. 3

E. None of the above



If we insert 7 into this binary **min**-heap, how many swaps will we need to do?

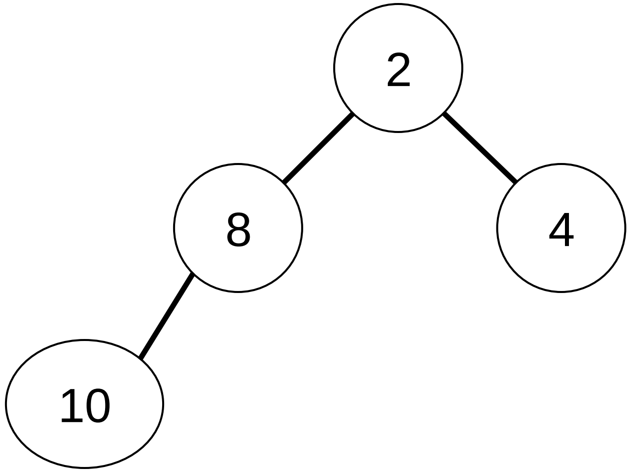
A. 0

B. 1

C. 2

D. 3

E. None of the above



What is the array representation of the following min-heap tree?

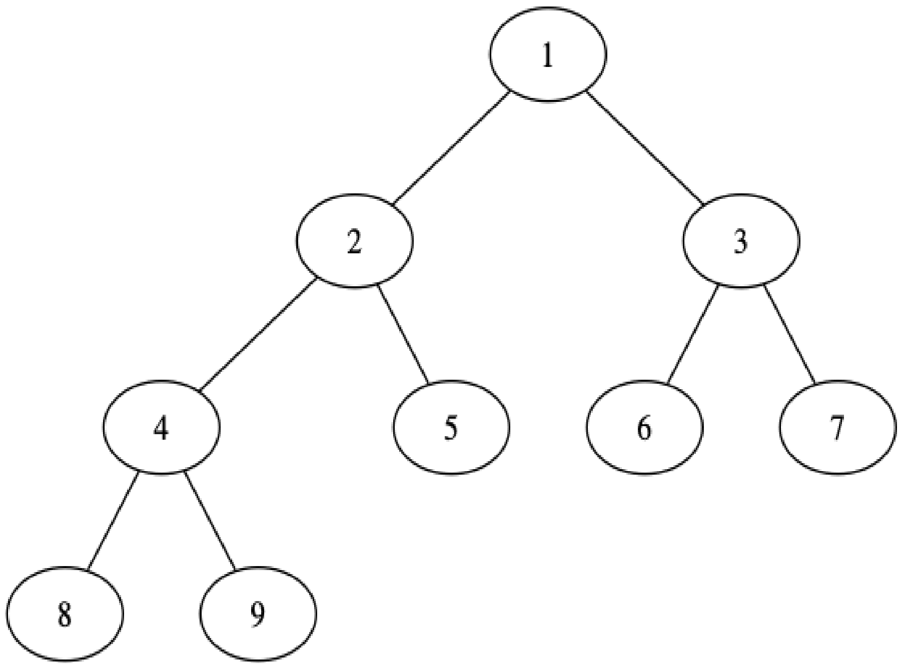
A. [8, 4, 9, 2, 5, 1, 6, 3, 7]

B. [1, 2, 3, 4, 5, 6, 7, 8, 9]

C. [1, 2, 4, 8, 9, 5, 3, 6, 7]

D. [8, 9, 4, 5, 2, 6, 7, 3, 1]

E. Something else



root = 0
left = 2 x i+1
right = 2 x i + 2

Priority Queue ADT: possible Data Structures

	top	enqueue	dequeue
Unsorted array/list	O(n)	O(1)	O(n)
Sorted array/list	O(1)	O(n)	O(1)
Balanced BST	O(log n)	O(log n)	O(log n)
Binary heap	O(1)	O(log n)	O(log n)

Priority Queue with binary heap: notes

- Binary heap can be used to implement *Priority Queue* ***ADT***

- Heap implementation is very efficient: all required update operations work in time $O(\log n)$

- Heap implementation as an array is also **space efficient**: we only store an array of priorities. Parent-child relationships are not stored, but are implied by the positions in the array

Common implementations of Priority Queues using Heaps

- C++: *priority_queue* in *std* library
- Java: *PriorityQueue* in *java.util* package
- Python: *heapq* (separate module)

Underneath is a Dynamic Array